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ON SPECIAL WEAKLY RICCI-SYMMETRIC S-MANIFOLD

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Abstract. In this paper, we have studied special weakly Ricci symmetric $S-$manifolds.

1. Introduction

In 1970, D.E Blair defined and studied $S-$manifolds which reduce, in special case, to normal contact manifolds [1]. Several authors have been studied on some properties of S- manifolds since then.

As a generalization of Chaki's pseudosymmetric and pseudo Ricci symmetric manifolds (see [4] and [5]), the notion of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L. Tamássy and T. Q. Binh (see [10] and [11]). These type manifolds were studied with in different structures by several authors (see [9], [10]). The notion of special weakly Ricci symmetric manifolds was introduced and studied by H. Singh, and Q. Khan in [8].

In this paper, we have studied some geometric properties of special weakly Ricci-symmetric $S-$manifolds. The paper is organized as follows. In section 2, we give a brief account of $S-$manifolds. In section 3, we consider a special weakly Ricci-symmetric $S-$manifold admits a cyclic parallel Ricci tensor and we show that under these conditions the $1-$form α must be vanish. On the other hand we show that the Ricci tensor of a special weakly Ricci-symmetric $S-$ manifold is parallel and we find the necessary condition for a special weakly Ricci-symmetric S- manifold to be an Einstein manifold.

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2. S-Manifolds

We need the following definition which is given in [6].

Let (M, g) be a Riemannian manifold with $dim(M) = 2n + s$ and denote by $T(M)$ the Lie algebra of vector field in M. Then M is said to be an $S-$Manifold if there exist on M an $f-$structure f [12] of rank $2n$ and s global vector fields $\xi_1, ..., \xi_s$(structure vector fields) such that [1]

(i) If $\eta_1, ..., \eta_s$ are dual $1-$forms of $\xi_1, ..., \xi_s$, then:

$$f\xi_i = 0 \quad \eta_i \circ f = 0 \quad f^2 = -I + \sum \xi_i \otimes \eta_i \tag{1}$$

$$g(X, Y) = g(fX, fY) + \sum \eta_i(X)\, \eta_i(Y) \tag{2}$$

for any $X, Y \in T(M)$, $i = 1, ..., s$.

(ii) The $f-$structure f is normal, that is

$$[f, f] + 2\sum \xi_i \otimes d\eta_i = 0$$

where $[f, f]$ is the Nijenhuis torsion of f.

(iii) $\eta_1 \wedge ... \wedge \eta_s \wedge (d\eta_i)^n \neq 0$ and $d\eta_1 = ... = d\eta_s = F$, for any i, where F is the fundamental $2-$form define by $F(X, Y) = g(X, fY)$, $X, Y \in T(M)$. In the case $s = 1$, an $S-$manifold is a Sasakian manifold. For $s \geq 2$, examples of $S-$manifolds are given in([2],[3]).

In a $S-$manifold M, besides the relations (1) and (2) the following also hold [7] :

$$(D_X f)\, Y = \sum_{i=1}^{n} \left[g(fX, fY)\xi_i + \eta_i(Y)\, f^2 X \right]$$

$$\eta_i(\xi_j) = \delta_{ij} \tag{3}$$

$$g(\xi_i, X) = \eta_i(X),\ 1 \leq i \leq n \tag{4}$$

$$Ric(\xi_i, X) = 2n \sum_{\beta=1}^{n} \eta_\beta(X) \tag{5}$$

$$D_X \xi_i = -fX$$

for any vector fields $X, Y \in T(M)$. Where Ric is Ricci tensor.

3. On special weakly Ricci-symmetric $S-$manifolds

An n-dimensional Riemannian manifold (M^n, g) is called a *special weakly Ricci-symmetric manifold* $(SWRS)_n$ if

(6) $(\nabla_X Ric)(Y, Z) = 2\alpha(X)Ric(Y, Z) + \alpha(Y)Ric(X, Z) + \alpha(Z)Ric(Y, X)$,

for any vector fields X, Y on M^n, where α is a $1-$form and is defined by

$$\alpha(X) = g(X, \rho), \tag{7}$$

where ρ is the associated vector field [8].

Theorem 3.1. *If a special weakly Ricci-symmetric S manifold admits a cyclic parallel Ricci tensor then the $1-$form α must be vanish.*

Proof. Let (6) and (7) be satisfied in a S manifold M. Taking cyclic sum of (6), we get

$$\begin{aligned}(\nabla_X Ric)(Y,Z)+(\nabla_Y Ric)(Z,X)+(\nabla_Z Ric)(X,Y)\\ =4(\alpha(X)Ric(Y,Z)+\alpha(Y)Ric(X,Z)+\alpha(Z)Ric(Y,X)).\end{aligned} \tag{8}$$

Let M^n admit a cyclic Ricci tensor. Then (8) reduces to

$$\alpha(X)Ric(Y,Z)+\alpha(Y)Ric(X,Z)+\alpha(Z)Ric(Y,X)=0. \tag{9}$$

Taking $Z=\xi_\alpha$ in (9) and using (5) and (7), we get

$$2n\sum\big[\alpha(X)\eta_\beta(Y)+\alpha(Y)\eta_\beta(X)\big]+\eta_\alpha(\rho)Ric(Y,X)=0. \tag{10}$$

Now putting $Y=\xi_\alpha$ in (10) and using (3), (5) and (7), we get

$$2n\alpha(X)+2n\sum\eta_\alpha(\rho)\eta_\beta(X)+\eta_\alpha(\rho)Ric(\xi_\alpha,X)=0. \tag{11}$$

Taking $X=\xi_\alpha$ in (11) and using (3), and (7), we get

$$\eta_\alpha(\rho)=0. \tag{12}$$

So by the use of (12) in (11), we have $\alpha(X)=0$, for any vector fields X on M. This completes the proof of the theorem. □

Theorem 3.2. *A special weakly Ricci-symmetric S manifold can not be an Einstein manifold if the $1-$form $\alpha\neq 0$.*

Proof. For an Einstein manifold $(\nabla_X Ric)(Y,Z)=0$ and $Ric(Y,Z)=kg(Y,Z)$, then (6) gives

$$2\alpha(X)Ric(Y,Z)+\alpha(Y)Ric(X,Z)+\alpha(Z)Ric(Y,X)=0. \tag{13}$$

Taking $Z=\xi_\alpha$ in (13) and (7), we get

$$2\alpha(X)Ric(Y,\xi_\alpha)+\alpha(Y)Ric(X,\xi_\alpha)+\alpha(\xi_\alpha)Ric(Y,X)=0. \tag{14}$$

Taking $X=\xi_\alpha$ in (14) and (7), we get

$$2\alpha(\xi_\alpha)Ric(Y,\xi_\alpha)+\alpha(Y)Ric(\xi_\alpha,\xi_\alpha)+\alpha(\xi_\alpha)Ric(Y,\xi_\alpha)=0. \tag{15}$$

Taking $Y=\xi_\alpha$ in (15) and using (3) and (7), we get

$$\eta_\alpha(\rho)=0. \tag{16}$$

Using (16) in (15), we get $\alpha(Y)=0$, for any vector fields Y on M^n, which completes the proof. □

Theorem 3.3. *The Ricci tensor of a special weakly Ricci-symmetric $S-$manifold is parallel.*

Proof. Taking $Z=\xi_\alpha$ in (6), we have

$$(\nabla_X Ric)(Y,\xi_\alpha)=2\alpha(X)Ric(Y,\xi_\alpha)+\alpha(Y)Ric(X,\xi_\alpha)+\alpha(\xi_\alpha)Ric(Y,X). \tag{17}$$

The left -hand side can be written in the form

$$(\nabla_X Ric)(Y,\xi_\alpha)=\nabla_X Ric(Y,\xi_\alpha)-Ric(\nabla_X Y,\xi_\alpha)-Ric(Y,\nabla_X\xi_\alpha). \tag{18}$$

Then, in view of (5), (7), and (18), equation (17) becomes

$$\begin{aligned}\nabla_X Ric(Y,\xi_\alpha)-Ric(\nabla_X Y,\xi_\alpha)-Ric(Y,\nabla_X\xi_\alpha)\\ =2\alpha(X)Ric(Y,\xi_\alpha)+\alpha(Y)Ric(X,\xi_\alpha)+\eta_\alpha(\rho)Ric(Y,X)\end{aligned} \tag{19}$$

Taking $Y = \xi_\alpha$ in (19) and using (7), we get

$$2\alpha(X)Ric(\xi_\alpha, \xi_\alpha) + \eta_\alpha(\rho)Ric(X, \xi_\alpha) + \eta_\alpha(\rho)Ric(\xi_\alpha, X) = 0 \tag{20}$$

Putting $X = \xi_\alpha$ in (20) we obtain

$$\eta_\alpha(\rho) = 0 \tag{21}$$

Using (21) in (20) we get

$$\alpha(X) = 0, \ \forall X \in T(M) \tag{22}$$

Hence in view of (22) in (6), we get $\nabla_X Ric = 0$, which is proves the result.
□

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