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SOME GEOMETRIC INEQUALITIES OF IONESCU-WEITZENBÖCK TYPE

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Abstract. Some Ionescu-Weitzenböck's type inequalities for general triangles and convex poligons are presented. We present a proof of Finsler-Hadwiger's inequality based on an algebraic identity and several geometric inequalities of Ionescu-Weitzenböck's type.

In memory of Ion Ionescu

"Give to Caesar what belongs to Caesar and to God what belongs to God."- Matthew 22:21

ION IONESCU (BIZET)
(1870 – 1946)

1. **Introduction**

D.M. Bătineţu-Giurgiu and N. Stanciu, demonstrated that the Weitzenböck's inequality must be named the Ionescu-Weitzenböck's inequality [1]. Their proof is based on a inequality of *Ion Ionescu*, the founder of Romanian Mathematical Gazette, appered in [5] as the contrapositive proposition of the following: If S is the area of a triangle ABC, then the following

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inequality holds

$$4S\sqrt{3} \leq a^2 + b^2 + c^2, \tag{1}$$

where a, b, and c are the side lengths of triangle ABC. A rigurous proof of the inequality (1) was given by N. Muzicescu [13] using certain geometrical constructions. The standard simple proof of (1) was given by I. Moscuna and I. Penescu [12] and they are based on the Law of Cosines.

In the year 1919 the inequality (1) was published by R. Weitzenböck, in [14], and therefore from this moment the inequality (1) must be named the inequality of Ionescu-Weitzenböck. The inequality of Ionescu–Weitzenböck, was given to solve at third IMO, Veszprém, Ungaria, 8-15 iulie 1961. For more details we refer to the monograph of A. Engel [4], and to the papers of D.M. Batineţu and N. Stanciu [1], N. Minculete and I. Bursuc [9].

In [10] is given the following inequality

$$a^2 + b^2 + c^2 \geq 4S\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right),$$

which implies the inequality of Ionescu – Weitzenböck, because $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq \sqrt{3}$.

A very important result is given in [14], where S.-H. Wu, Z.-H. Zhang and Z.-G. Xiao proved that Ionescu-Weitzenböck's inequality and Finsler-Hadwiger's inequality,

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S + (a-b)^2 + (b-c)^2 + (c-a)^2,$$

are equivalent, i.e. applying Ionescu-Weitzenböck's Inequality in a special triangle, we deduce Finsler-Hadwiger's inequality.

C. Lupu, R. Marinescu and S. Monea in [7] treated above inequality and A. Cipu [3] shows optimal reverse Finsler-Hadwiger inequalities.

The more general form of (1) appeared in [8], as follow:

$$S \leq \frac{\sqrt{3}}{4}\left(\frac{a^k + b^k + c^k}{3}\right)^{\frac{2}{k}}, k > 0.$$

Inequality (1) was generalized for a convex polygon by E. Just and N. Schaumberger's [6], thus: Let $A_1A_2...A_n$, $n \geq 3$, be a convex polygon with a_k the length of the side $[A_kA_{k+1}]$, $k = \overline{1, n}$, $A_{n+1} = A_1$,and let S be the area of the polygon. Then, we have that:

$$\sum_{k=1}^{n} a_k^2 \geq 4S \cdot \tan\frac{\pi}{n}. \tag{2}$$

2. **Main results**

As mentioned above there are numerous demonstrations of Ionescu-Weitzenböck's inequality and of Finsler-Hadwiger's inequality. Knowing that these two inequalities are equivalent, now we present a proof of Finsler-Hadwiger's inequality based on an identity.

Proposition 2.1. *In any triangle there is the following identity:*

$$\left[\sum_{cyc} a^2 - \sum_{cyc}(a-b)^2\right]^2 = \left(4\sqrt{3}S\right)^2 + 8\sum_{cyc}(s-c)^2(a-b)^2. \tag{3}$$

Proof. It is known the following algebraic identity:

$$(x+y+z)^2 = 3(xy+yz+zx) + \frac{1}{2}\left[(x-y)^2+(y-z)^2+(z-x)^2\right], \tag{4}$$

where x, y, z are real numbers. In relation (4), if we take $x=(s-b)(s-c)$,$y=(s-a)(s-c)$ and $z=(s-a)(s-b)$, then we obtain the following identity:

$$\left(\sum_{cyc}(s-b)(s-c)\right)^2 = 3\sum_{cyc}(s-a)(s-b)(s-c)^2 + \frac{1}{2}\sum_{cyc}(s-c)^2(a-b)^2.$$

But, we have

$$\sum_{cyc}(s-b)(s-c) = \sum_{cyc}[s^2 - s(b+c)+bc)] = ab+bc+ca-s^2$$

$$= ab+bc+ca - \frac{(a+b+c)^2}{4} = \frac{1}{4}\left[\sum_{cyc}a^2 - \sum_{cyc}(a-b)^2\right]$$

and

$$\sum_{cyc}(s-a)(s-b)(s-c)^2 = (s-a)(s-b)(s-c)\sum_{cyc}(s-c) = S^2.$$

Therefore, we deduce the identity

$$\frac{1}{16}\left[\sum_{cyc}a^2 - \sum_{cyc}(a-b)^2\right]^2 = 3S^2 + \frac{1}{2}\sum_{cyc}(s-c)^2(a-b)^2,$$

which implies the statement. □

Remark 2.1. *It is easy to see that* $8\sum_{cyc}(s-c)^2(a-b)^2 \geq 0$*, which means that relation* (3) *becomes* $\sum_{cyc}a^2 - \sum_{cyc}(a-b)^2 \geq 4\sqrt{3}S$*, which is Finsler-Hadwiger's inequality.*

In the next we present several applications of Ionescu-Weitzenböck's inequality given by following results:

Proposition 2.2. *If* $m \in R_+, x, y \in R_+^*$*, then in anyl triangle* ABC *holds:*

$$\frac{a^{m+2}}{(xb+yc)^m} + \frac{b^{m+2}}{(xc+ya)^m} + \frac{c^{m+2}}{(xa+yb)^m} \geq \frac{4\sqrt{3}}{(x+y)^m}\cdot S.$$

Proof. By Radon's inequality we deduce that:

$$E = \sum_{cyc}\frac{a^{m+2}}{(xb+yc)^m} = \sum_{cyc}\frac{a^{2(m+1)}}{(xab+yac)^m} \geq \frac{(a^2+b^2+c^2)^{m+1}}{\left(\sum_{cyc}(xab+yac)\right)^m} =$$

$$= \frac{(a^2+b^2+c^2)^{m+1}}{(x+y)^m(ab+bc+ca)^m},$$

and taking account that $a^2+b^2+c^2 \geq ab+bc+ca$, we deduce:

$$E \geq \frac{(a^2+b^2+c^2)^{m+1}}{(x+y)^m(a^2+b^2+c^2)^m} = \frac{a^2+b^2+c^2}{(x+y)^m}. \tag{5}$$

By (5) and (1) we obtain the conclusion. □

Proposition 2.3. *In any triangle ABC the following inequality holds:*

$$\left(a^6+b^6+c^6\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq 48S^2. \tag{6}$$

Proof. By Radon's inequality we deduce that:

$$\sum a^6 = \sum \left(a^2\right)^3 \geq \frac{\left(\sum a^2\right)^3}{3^2} \tag{7}$$

and by Bergström's inequality we obtain that:

$$\sum \frac{1}{a^2} \geq \frac{3^2}{\sum a^2} \tag{8}$$

By (7) and (8) yields that:

$$U = \left(\sum a^6\right)\left(\sum \frac{1}{a^2}\right) \geq \left(\sum a^2\right)^2 \tag{9}$$

By (1) and (9) we obtain that $U \geq \left(4S\sqrt{3}\right)^2 = 48S^2$, and we are done. □

Proposition 2.4. *In any triangle ABC the following inequality holds:*

$$\left(a^4+b^4+c^4\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq 12\sqrt{3}S. \tag{10}$$

Proof. By Bergström's inequality we deduce that:

$$\sum a^4 = \sum \left(a^2\right)^2 \geq \frac{\left(\sum a^2\right)^2}{3} \tag{11}$$

and

$$\sum \frac{1}{a^2} \geq \frac{3^2}{\sum a^2}. \tag{12}$$

So, (11) and (12) yields that:

$$U = \left(\sum a^4\right)\left(\sum \frac{1}{a^2}\right) \geq 3\sum a^2 \tag{13}$$

By (1) and (13) we obtain that: $U \geq 3 \cdot \left(4S\sqrt{3}\right) = 12\sqrt{3}S$, and we are done. □

Proposition 2.5. *Let $A_1A_2...A_n$, $n \geq 3$, be a convex polygon with a_k the length of the side $[A_kA_{k+1}]$, $k=\overline{1,n}$, $A_{n+1}=A_1$, and let S be the area of the polygon. Then we have that:*

$$\left(\sum_{k=1}^{n} a_k^{2m+4}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k^{2m}}\right) \geq 16S^2 tan^2\frac{\pi}{n}, \forall m \in R_+. \tag{14}$$

Proof. By Radon's inequality we deduce that:

$$\sum_{k=1}^{n} a_k^{2m+4} \geq \frac{1}{n^{m+1}} \left(\sum_{k=1}^{n} a_k^2 \right)^{m+2}, \forall m \in R_+ \tag{15}$$

and

$$\sum_{k=1}^{n} \frac{1}{a_k^{2m}} \geq \frac{n^{m+1}}{\left(\sum_{k=1}^{n} a_k^2\right)^m}, \forall m \in R_+ \tag{16}$$

So, (15) and (16) yields that:

$$U_n = \left(\sum_{k=1}^{n} a_k^{2m+4} \right) \left(\sum_{k=1}^{n} \frac{1}{a_k^{2m}} \right) \geq \left(\sum_{k=1}^{n} a_k^2 \right)^2, \forall m \in R_+ \tag{17}$$

Therefore, from (17) and (2) we obtain that: $U_n \geq 16S^2 \tan^2 \frac{\pi}{n}, \forall m \in R_+$ and we are done. □

Proposition 2.6. *If m is a real positive number and n is a positive integer number, and if ABC, is a triangle with usual notations, then:*

(18)
$$\frac{a^{2(m+1)}}{\left(F_n m_a^2 + F_{n+1} m_b^2\right)^m} + \frac{b^{2(m+1)}}{\left(F_n m_b^2 + F_{n+1} m_c^2\right)} + \frac{c^{2(m+1)}}{\left(F_n m_c^2 + F_{n+1} m_a^2\right)} \geq \frac{4^{m+1}\sqrt{3}}{3^m F_{n+2}^m} S,$$

$$\frac{m_a^{2(m+1)}}{\left(F_n b^2 + F_{n+1} c^2\right)^m} + \frac{m_b^{2(m+1)}}{\left(F_n c^2 + F_{n+1} a^2\right)} + \frac{m_c^{2(m+1)}}{\left(F_n a^2 + F_{n+1} b^2\right)} \geq \frac{3^{m+1}\sqrt{3}}{4^m F_{n+2}^m} S, \tag{19}$$

$$\frac{a^{2(m+1)}}{\left(F_n m_a^2 + F_{n+1} m_b^2 + F_{n+2} m_c^2\right)^m} + \frac{b^{2(m+1)}}{\left(F_n m_b^2 + F_{n+1} m_c^2 + F_{n+2} m_a^2\right)} + \frac{c^{2(m+1)}}{\left(F_n m_c^2 + F_{n+1} m_a^2 + F_{n+2} m_b^2\right)} \geq \frac{2^{m+2}\sqrt{3}}{3^m F_{n+2}^m} S \tag{20}$$

and

$$\frac{m_a^{2(m+1)}}{\left(F_n a^2 + F_{n+1} b^2 + F_{n+2} c^2\right)^m} + \frac{m_b^{2(m+1)}}{\left(F_n b^2 + F_{n+1} c^2 + F_{n+2} a^2\right)} + \frac{m_c^{2(m+1)}}{\left(F_n c^2 + F_{n+1} a^2 + F_{n+2} b^2\right)} \geq \frac{3^{m+1}\sqrt{3}}{8^m F_{n+2}^m} S, \tag{21}$$

where F_n is the sequences of Fibonacci.

Proof. In any triangle is known the formula

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right), \tag{22}$$

where m_a, m_b, m_c are the lengths of the medians. We observe that

$$I_n = \sum \frac{a^{2(m+1)}}{\left(F_n m_a^2 + F_{n+1} m_b^2\right)^m} = \sum \frac{\left(a^2\right)^{m+1}}{\left(F_n m_a^2 + F_{n+1} m_b^2\right)^m},$$

Using Radon's inequality, we have

$$I_n \geq \frac{\left(a^2 + b^2 + c^2\right)^{m+1}}{\left(\sum \left(F_n m_a^2 + F_{n+1} m_b^2\right)\right)^m} = \frac{\left(a^2 + b^2 + c^2\right)^{m+1}}{\left(F_n + F_{n+1}\right)^m \left(m_a^2 + m_b^2 + m_c^2\right)^m} =$$

$$= \frac{\left(a^2+b^2+c^2\right)^{m+1}}{F_{n+2}^m\left(m_a^2+m_b^2+m_c^2\right)^m}.$$

But, from relation (22), we deduce the relation $I_n \geq \frac{4^m}{3^m F_{n+2}^m}\left(a^2+b^2+c^2\right)$. By (1), we obtain that $I_n \geq \frac{4^{m+1}\sqrt{3}}{3^m F_{n+2}^m} S$. So, relation (18) is proved. Now, we have the equality

$$Y_n = \sum \frac{m_a^{2(m+1)}}{\left(F_n b^2 + F_{n+1} c^2\right)^m} = \sum \frac{\left(m_a^2\right)^{m+1}}{\left(F_n b^2 + F_{n+1} c^2\right)^m},$$

where applying Radon 's inequality, we deduce that

$$Y_n \geq \frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{\left(\sum\left(F_n b^2+F_{n+1}c^2\right)\right)^m} = \frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{\left(F_n+F_{n+1}\right)^m\left(a^2+b^2+c^2\right)^m} =$$

$$= \frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{F_{n+2}^m\left(a^2+b^2+c^2\right)^m}.$$

But, using the relation (22), yields $Y_n \geq \frac{3^{m+1}}{4^{m+1}F_{n+2}^m}\left(a^2+b^2+c^2\right)$. By (1) we deduce that $Y_n \geq \frac{3^{m+1}\sqrt{3}}{4^m F_{n+2}^m} S$, thus relation (19) is proved. We also have that

$$Z_n = \sum \frac{a^{2(m+1)}}{\left(F_n m_a^2 + F_{n+1} m_b^2 + F_{n+2} m_c^2\right)^m} = \sum \frac{\left(a^2\right)^{m+1}}{\left(F_n m_a^2 + F_{n+1} m_b^2 + F_{n+2} m_c^2\right)^m},$$

and from the inequality of Radon we deduce that

$$Z_n \geq \frac{\left(a^2+b^2+c^2\right)^{m+1}}{\left(\sum\left(F_n m_a^2 + F_{n+1} m_b^2 + F_{n+2} m_c^2\right)\right)^m} = \frac{\left(a^2+b^2+c^2\right)^{m+1}}{\left(F_n+F_{n+1}+F_{n+2}\right)^m\left(m_a^2+m_b^2+m_c^2\right)^m} =$$

$$= \frac{\left(a^2+b^2+c^2\right)^{m+1}}{2^m F_{n+2}^m\left(m_a^2+m_b^2+m_c^2\right)^m}.$$

Because, using equality (22), we have

$$Z_n \geq \frac{2^m}{3^m F_{n+2}^m}\left(a^2+b^2+c^2\right).$$

By (1) we obtain that $Z_n \geq \frac{2^{m+2}\sqrt{3}}{3^m F_{n+2}^m} S$. So, relation (20) is proved. We have

$$X_n = \sum \frac{m_a^{2(m+1)}}{\left(F_n a^2 + F_{n+1} b^2 + F_{n+2} c^2\right)^m} = \sum \frac{\left(m_a^2\right)^{m+1}}{\left(F_n a^2 + F_{n+1} b^2 + F_{n+2} c^2\right)^m},$$

and from Radon's inequality we obtain that

$$X_n \geq \frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{\left(\sum\left(F_n a^2+F_{n+1}b^2+F_{n+2}c^2\right)\right)^m} = \frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{\left(F_n+F_{n+1}+F_{n+2}\right)^m\left(a^2+b^2+c^2\right)^m} =$$

$$= \frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{2^m F_{n+2}^m\left(a^2+b^2+c^2\right)^m}.$$

From identity (22), we deduce:

$$X_n \geq \frac{3^{m+1}}{2^{3m+2}F_{n+2}^m}\left(a^2+b^2+c^2\right).$$

Then by (1), follows that

$$X_n \geq \frac{3^{m+1}\sqrt{3}}{2^{3m}F_{n+2}^m}S,$$

and the proof is complete. □

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