A NEW NEUBERG-PEDOE TYPE INEQUALITY FOR TWO TETRAHEDRONS WITH APPLICATIONS

WANG WEN, YANG SHI-GUO, YU JING, QI JI-BING

Abstract. In this paper, the theory and method of distance geometry is used to study the problems of geometric inequalities for the product of the opposite edges and the volumes of two tetrahedrons in the 3-dimensional Euclidean space $E^3$. We shall present a new version of the Neuberg-Pedoe type inequality involving the volumes and the product of opposite edges of two tetrahedrons. In addition, by applying the main Theorem, we will present a new version of the Finsler-Hadwiger type inequality of tetrahedron, and establish a generalization and strengthening of Euler inequality of tetrahedron.

1. Introduction

We denote by $a_i, b_i, c_i$ the edge lengths of the triangle $A_iB_iC_i$ ($i = 1, 2$) with the area $\Delta_i$. The famous Neuberg-Pedoe inequality reads as follows:

$$H_2 = a_1^2(b_2^2 + c_2^2 - a_2^2) + b_1^2(a_2^2 + c_2^2 - b_2^2) + c_1^2(a_2^2 + b_2^2 - c_2^2) \geq 16\Delta_1\Delta_2,$$

with equality holds if and only if two triangle are similar ([11]).

In 1984, P. Chiakuei proved the following sharpening of Neuberg-Pedoe inequality ([5]).

$$H_2 \geq 8\left(\frac{a_1^2 + b_1^2 + c_1^2}{a_2^2 + b_2^2 + c_2^2} \Delta_2 + \frac{a_2^2 + b_2^2 + c_2^2}{a_1^2 + b_1^2 + c_1^2} \Delta_1\right),$$

with equality holds if and only if two triangle are similar.

The Neuberg-Pedoe inequality (1) has attracted the interest of many mathematicians and has motivated a large number of research papers involving different proofs, various generalizations, improvements and applications (see [9]).

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Recently, Chen and Wang [1], Tang and Leng [6] generalized inequality (1) to two tetrahedrons, and established the following two inequalities, respectively.

We denote by \( a_i \) \((i = 1...6)\) the edge lengths and by \( F_i \) \((i = 1...4)\) the areas of the \(i\)-th facet of the tetrahedron \(A_1A_2A_3A_4\), with the volume \(V\). For another tetrahedron \(A'_1A'_2A'_3A'_4\), we use the analogous notations. For example, \(V'\) the volume of the tetrahedron \(A'_1A'_2A'_3A'_4\). Then

\[
\begin{align*}
3 \sum_{i=1}^{4} F'_i \left( \frac{4}{3} F_i - 2F'_i \right) &\geq 18 \sqrt[3]{3} (VV')^{2/3},\hspace{2cm} (3) \\
\sum_{i=1}^{6} a_i \left( \sum_{j=1}^{6} a_j - 2a_i \right) &\geq 24 \cdot 72^\frac{\theta}{2} (VV')^\frac{\theta}{2}, \hspace{2cm} (4)
\end{align*}
\]

with equalities hold if and only if two tetrahedrons are regular, where \(\theta \in (0,1]\).

For a long time, many researchers have studied the relationship between the tetrahedral elements by using different methods, and established a large number of geometric inequalities ([3], [7], [10], [15], [17]). In 1989, D.S. Mitrinović, J.E. Pečarić and V. Volenec described in detail in their work “Recent Advances in Geometric Inequalities” lots of geometric inequalities of tetrahedron ([9]). On this basis, we further discuss the problems of geometric inequalities of tetrahedron.

This paper, except for the introduction, is divided into three sections. In section 2, we will present a new version of the Neuberg-Pedoe type inequality involving the volumes and the product of opposite edges of two tetrahedron. In section 3, some lemmas and the proof of main theorem will be given. Finally, by applying the Theorem 2.2, we will present a new version of the Finsler-Hadwiger type inequality of tetrahedron, and establish generalization and strengthening of Euler inequality of tetrahedron.

2. MAIN RESULT

We first establish some notations: Let \(A_1B_1C_1D_1\) and \(A_2B_2C_2D_2\) be two tetrahedrons in \(E^3\) with the volumes \(V\) and \(V'\), respectively. Let \(a_i = B_iC_i\), \(b_i = C_iA_i\), \(c_i = A_iB_i\), \(a'_i = A_iD_i\), \(b'_i = B_iD_i\), \(c'_i = C_iD_i\) \((i = 1,2)\) be the edge lengths of the tetrahedrons \(A_1B_1C_1D_1\). Therefore, \(a_i\) and \(a'_i\), \(b_i\) and \(b'_i\), \(c_i\) and \(c'_i\) are three groups opposite edges of tetrahedron, respectively.

**Theorem 2.1.** For any two tetrahedrons \(A_1B_1C_1D_1\) and \(A_2B_2C_2D_2\), if \(0 < \theta \leq 1\), we have the following inequality

\[
H = (a_1a'_1)^\theta \left[ \left( a_2b'_2 \right)^\theta + \left( c_2c'_2 \right)^\theta - \left( a_2a'_2 \right)^\theta \right] + (b_1b'_1)^\theta \left[ \left( a_2a'_2 \right)^\theta + \left( c_2c'_2 \right)^\theta - \left( b_2b'_2 \right)^\theta \right] + \hspace{2cm} (5)
\]

\[
+ (c_1c'_1)^\theta \left[ \left( a_2a'_2 \right)^\theta + \left( b_2b'_2 \right)^\theta - \left( c_2c'_2 \right)^\theta \right] \geq 3 \cdot 4^\theta \cdot \frac{10^\theta}{\pi} (VV')^{\frac{2\theta}{3}},
\]

with equality hold if two tetrahedrons are regular.

In fact, we can established an inequality more general than (5), as follows.
Theorem 2.2. For any two tetrahedrons $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$, if $0 < \theta \leq 1$, we have the following inequality

$$H \geq \frac{3}{2} \cdot 4^{\theta} \cdot 3^{\frac{4\theta}{3}} \left[ \frac{\sigma'}{\sigma} V^{\frac{4\theta}{3}} + \frac{\sigma}{\sigma'} V^{\frac{4\theta}{3}} \right],$$

with equality hold if two tetrahedrons are regular, where

$$\sigma = (a_1a'_1)^\theta + (b_1b'_1)^\theta + (c_1c'_1)^\theta;$$

$$\sigma' = (a_2a'_2)^\theta + (b_2b'_2)^\theta + (c_2c'_2)^\theta.$$

In addition, taking $\theta = 1$ in Theorem 2.2 and using the arithmetic-geometric mean inequality, we get the following Corollary.

Corollary 2.1. For any two tetrahedrons $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$, we have

$$a_1a'_1 (b_2b'_2 + c_2c'_2 - a_2a'_2) + b_1b'_1 (a_2a'_2 + c_2c'_2 - b_2b'_2) + c_1c'_1 (a_2a'_2 + b_2b'_2 - c_2c'_2)$$

$$\geq 18\sqrt{3} \left[ \frac{a_2a'_2 + b_2b'_2 + c_2c'_2 V^{\frac{2}{3}}}{a_1a'_1 + b_1b'_1 + c_1c'_1} + \frac{a_1a'_1 + b_1b'_1 + c_1c'_1 V^{\frac{2}{3}}}{a_2a'_2 + b_2b'_2 + c_2c'_2} \right]$$

$$\geq 36\sqrt{3}VV' \frac{2}{3}$$,

with equality hold if two tetrahedrons are regular.

3. SOME LEMMAS AND THE PROOF OF THE THEOREM 2.2

To prove Theorem 2.2 in section 2, we need some lemmas as follows.

Lemma 3.1. For any tetrahedron $A_1B_1C_1D_1$ and $\theta \in (0, 1]$, we have

$$\begin{aligned}
(a_1a'_1)^\theta + (b_1b'_1)^\theta > (c_1c'_1)^\theta, \\
(b_1b'_1)^\theta + (c_1c'_1)^\theta > (a_1a'_1)^\theta, \\
(a_1a'_1)^\theta + (c_1c'_1)^\theta > (b_1b'_1)^\theta.
\end{aligned}$$

Proof. The following inequalities are known (see [9, p.549])

$$\begin{aligned}
a_1a'_1 + b_1b'_1 > c_1c'_1, \\
b_1b'_1 + c_1c'_1 > a_1a'_1, \\
a_1a'_1 + c_1c'_1 > b_1b'_1.
\end{aligned}$$

By applying inequality $x^{\frac{2}{\theta}} \leq x$ ($0 < x < 1, \theta \in (0, 1]$) and (10), we can infer (9). $\square$

Lemma 3.2 (see [9, p.556]). For any tetrahedron $A_1B_1C_1D_1$ and $\theta \in (0, 1]$, we have

$$\begin{aligned}
a_1b_1c_1a'_1b'_1c'_1 \geq 72V^2, \\
(b_1b'_1 + c_1c'_1)(a_1a'_1 + c_1c'_1 - b_1b'_1)(a_1a'_1 + b_1b'_1 - c_1c'_1) \geq 72V^2,
\end{aligned}$$

with equalities hold if $A_1B_1C_1D_1$ is regular.
Lemma 3.3 (see [2], [8]). Let $0 < \lambda_i < \frac{1}{2}$, $\sum_{i=1}^{n+1} \lambda_i = 1$, $x_i = \frac{1}{\lambda_i} - 2$ ($i = 1 \ldots n+1$). Then

$$\sum_{i=1}^{n+1} \left( \prod_{j=1}^{n+1} x_j \right) = \sum_{i=1}^{n+1} \prod_{j=1, j\neq i}^{n+1} \left( \frac{1}{\lambda_i} - 2 \right) \geq (n+1)(n-1)^n,$$

with equalities hold if and only if $\lambda_1 = \lambda_2 = \cdots \lambda_{n+1}$.

Lemma 3.4 (see [15]). Let $\Omega$ be an $n$-simplex in $E^n$ with the $n$-dimensional volume $V$, and $F_i$ the $(n-1)$-dimensional volumes of $(n-1)$-dimensional subsimplices spanned by $n-1$ vertexes $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{n+1}$ of $\Omega$ ($i = 1 \ldots n+1$). If $x_i \in R^+$. Then

$$\left( \sum_{i=1}^{n+1} x_i \right)^n \prod_{j=1}^{n+1} F_j^2 \geq \frac{\eta^{3a}}{(n!)^2} \sum_{i=1}^{n+1} \left( \prod_{j=1, j\neq i}^{n+1} x_j \right) F_i^2 \cdot V^{2(n-1)},$$

with equality holds if $\Omega$ is regular and $x_1 = x_2 = \cdots x_{n+1}$.

Lemma 3.5. Let $A_1B_1C_1D_1$ be a tetrahedron in $E^3$, $x_1, x_2, x_3$ denote any 3 positive real number. Then

$$\left( \sum_{i=1}^{3} x_i \right)^2 \prod_{j=1}^{3} K_j^2 \geq 432 \cdot 3^2 \cdot V^3 \sum_{i=1}^{3} \left( \prod_{j=1, j\neq i}^{3} x_j \right) K_i^2,$$

with equality holds if $A_1B_1C_1D_1$ is regular and $x_1 = x_2 = x_3$, where $K_1 = a_1 a'_1$, $K_2 = b_1 b'_1$, $K_3 = c_1 c'_1$.

Proof. From (10), we know that there exit a triangle $EFG$ with the edge lengths $a_1 a'_1, b_1 b'_1$ and $c_1 c'_1$. We denote by $\Delta$ the area of triangle $EFG$, and put $K_1 = a_1 a'_1$, $K_2 = b_1 b'_1$, $K_3 = c_1 c'_1$. Thus, for the triangle $EFG$, applying (11), we get

$$\left( \sum_{i=1}^{3} x_i \right)^2 \prod_{j=1}^{3} K_j^2 \geq 16 \Delta^2 \sum_{i=1}^{3} \left( \prod_{j=1, j\neq i}^{3} x_j \right) K_i^2,$$

with equality holds if $EFG$ is regular and $x_1 = x_2 = x_3$.

On the other hand, by Heron’s formula, we have

$$\Delta^2 = p(p-a_1 a'_1)(p-b_1 b'_1)(p-c_1 c'_1),$$

where $p = \frac{1}{2}(a_1 a'_1 + b_1 b'_1 + c_1 c'_1)$.
Applying (17), (10), (11) and the arithmetic-geometric mean inequality, we get
\[
\Delta^2 = p(p - a_1'c_1')(p - b_1'c_1)(p - c_1'c_1')
\]
\[
= \frac{a_1a_1' + b_1b_1' + c_1c_1' - a_1a_1' - b_1b_1' - c_1c_1'}{2} \cdot \frac{a_1a_1'c_1c_1' - b_1b_1'c_1c_1'}{2} \cdot \frac{a_1a_1' + b_1b_1' - c_1c_1'}{2}
\]
\[
\geq \frac{3}{16} \sqrt[3]{a_1b_1c_1a_1'b_1'c_1'} \cdot 72V^2
\]
i.e.
\[
\Delta^2 \geq 27 \cdot \frac{2^3}{3} V^8.
\]
(18)

By (18) and (16), we obtain (15).

\[\square\]

**Lemma 3.6.** Under the hypothesis in Lemma 3.5, if \( \theta \in (0, 1] \), we have
\[
\left( \sum_{i=1}^{3} x_i K_i^{2\theta} \right)^{2} \geq 9 \cdot 16^{\theta} \cdot \frac{88}{3} V^{\frac{88}{3}},
\]
(19)
with equality holds if \( A_1B_1C_1D_1 \) is regular and \( x_1 = x_2 = x_3 \).

**Proof.** By Maclaurin inequality [4], we get
\[
\left( \sum_{i=1}^{3} x_i \right)^{2} \geq 3 \left( \sum_{i=1}^{3} \prod_{j=1, j \neq i}^{3} x_j \right),
\]
(20)
with equality holds if and only if \( x_1 = x_2 = x_3 \).

Thus, employing Lemma 3.3, inequality (20) and Holder inequality, we have
\[
\left( \sum_{i=1}^{3} x_i \right)^{2} \prod_{j=1}^{3} k_j^{2\theta} = \left( \sum_{i=1}^{3} x_i \right)^{2(1-\theta)} \left[ \left( \sum_{i=1}^{3} x_i \right)^{2} \prod_{j=1}^{3} k_j^{2\theta} \right]^{\theta}
\]
\[
\geq 3^{1-\theta} \left[ \sum_{i=1}^{3} \prod_{j=1, j \neq i}^{3} x_j \right]^{1-\theta} \left[ 432 \times 3^{3} \right]^{\theta} \left[ \sum_{i=1}^{3} \prod_{j=1, j \neq i}^{3} x_j \right]^{\theta} K_i^{2\theta}
\]
\[
\geq 3^{1-\theta} (432 \times 3^{3})^{\theta} V^{\frac{88}{3}} \sum_{i=1}^{3} \prod_{j=1, j \neq i}^{3} x_j \cdot K_i^{2\theta}.
\]

By replacing \( x_i \) with \( x_i K_i^{2\theta} \) in above inequality, we get
\[
\left( \sum_{i=1}^{3} x_i K_i^{2\theta} \right)^{2} \geq 3 \cdot 16^{\theta} \cdot \frac{88}{3} V^{\frac{88}{3}} \sum_{i=1}^{3} \prod_{j=1, j \neq i}^{3} x_j.
\]
(21)
Taking \( n = 2 \) in the inequality (13), we get

\[
(22) \quad \sum_{i=1}^{3} \left( \prod_{j=1 \atop j \neq i}^{3} x_j \right) \geq 3.
\]

By inequalities (21) and (22), we get (19).

\[\square\]

**Lemma 3.7.** Under the hypothesis in Lemma 3.5, if \( \theta \in (0, 1] \), we have

\[
(23) \quad \left( \sum_{i=1}^{3} K_i^\theta \right)^2 - 2 \sum_{i=1}^{3} K_i^{2\theta} \geq 3 \cdot 4^\theta \cdot 3 \cdot \frac{4^\theta}{3^3} \, V \frac{4^\theta}{3^3},
\]

with equality holds if \( A_1B_1C_1D_1 \) is regular.

**Proof.** Taking \( x_i = \frac{K_1^\theta + K_2^\theta + K_3^\theta - 2K_i^\theta}{K_i^\theta} > 0 \) (\( i = 1, 2, 3 \)) in Lemma 3.6, we can infer (23).

\[\square\]

**Proof of Theorem 2.2.** Note

\[
K_1 = a_1a'_1, \quad K_2 = b_1b'_1, \quad K_3 = c_1c'_1, \\
S_1 = a_2a'_2, \quad S_2 = b_2b'_2, \quad S_3 = c_2c'_2,
\]

\[
T_1 = \left( \sum_{j=1}^{3} K_j^\theta \right)^2 - 2 \sum_{i=1}^{3} K_i^{2\theta},
\]

\[
T_2 = \left( \sum_{j=1}^{3} S_j^\theta \right)^2 - 2 \sum_{i=1}^{3} S_i^{2\theta}.
\]

By computation and applying Lemma 3.7, we have

\[
H - \frac{3}{2} \cdot 4^\theta \cdot 3 \cdot \frac{4^\theta}{3^3} \left[ \frac{\sigma'}{\sigma} V \frac{4^\theta}{3^3} + \frac{\sigma'}{\sigma'} V' \frac{4^\theta}{3^3} \right]
\]

\[
= \sum_{i=1}^{3} K_i^\theta \left( \sum_{j=1}^{3} S_j^\theta - 2S_i^{2\theta} \right) - \frac{3}{2} \cdot 4^\theta \cdot 3 \cdot \frac{4^\theta}{3^3} \left[ \frac{\sigma'}{\sigma} V \frac{4^\theta}{3^3} + \frac{\sigma'}{\sigma'} V' \frac{4^\theta}{3^3} \right]
\]

\[
= \frac{1}{2} \left[ \frac{\sigma'}{\sigma} T_1 + \frac{\sigma}{\sigma'} T_2 \right] + \frac{1}{\sigma \sigma'} \sum_{i=1}^{3} \left[ \sigma_i^{\theta} - \sigma S_i^{\theta} \right]^2 - \frac{3}{2} \cdot 4^\theta \cdot 3 \cdot \frac{4^\theta}{3^3} \left[ \frac{\sigma'}{\sigma} V \frac{4^\theta}{3^3} + \frac{\sigma'}{\sigma'} V' \frac{4^\theta}{3^3} \right]
\]

\[
\geq \frac{1}{\sigma \sigma'} \sum_{i=1}^{3} \left[ \sigma_i^{\theta} - \sigma S_i^{\theta} \right]^2
\]

\[
\geq 0.
\]

\[\square\]
4. Some Applications

For a triangle ABC with sides a, b, c and area Δ. The following inequalities are known [6].

\[(24)\] \[a + b + c \geq 2\sqrt[3]{27\sqrt{\Delta}},\]

\[(25)\] \[a^2 + b^2 + c^2 \geq 4\sqrt{3\Delta} + (a - b)^2 + (b - c)^2 + (c - a)^2,\]

with equalities hold if and only if triangle ABC is regular.

Inequality (25) is the well-known Finsler-Hadwiger’s inequality. In addition, inequalities (24) and (25) have been studied by many researchers, and some generalizations and improvements have been given in [13] and [14]. Following we shall give a new generalization of (24) and (25) for a tetrahedron.

In fact, taking tetrahedron \(A_1B_1C_1D_1 = A_2B_2C_2D_2\) in Theorem 2.1, we get the following theorem.

**Theorem 4.1.** Let ABCD be a tetrahedron in \(E^3\) with the volume \(V\). Let \(a = BC\), \(b = CA\), \(c = AB\), \(a' = AD\), \(b' = BD\), \(c' = CD\) be the edge lengths of the tetrahedrons ABCD. If \(\theta \in (0, 1]\), we have

\[(26)\] \[(aa')^{2\theta} + (bb')^{2\theta} + (cc')^{2\theta} \geq 3 \cdot 4^{\theta} \cdot 3^{\frac{2\theta}{3}} \cdot V^{\frac{2\theta}{3}} + [(aa')^{\theta} - (bb')^{\theta}]^2 + [(bb')^{\theta} - (cc')^{\theta}]^2 + [(cc')^{\theta} - (aa')^{\theta}]^2,\]

with equality holds if ABCD is regular.

Taking \(\theta = 1\) or \(\theta = \frac{1}{2}\) in inequality (26), we get the following corollary.

**Corollary 4.1.** With the same conditions as in Theorem 4.1, we have

\[(27)\] \[aa' + bb' + cc' \geq 6\sqrt[3]{9V^{\frac{1}{2}}} + [(aa')^{\frac{1}{2}} - (bb')^{\frac{1}{2}}] + [(bb')^{\frac{1}{2}} - (cc')^{\frac{1}{2}}] + [(cc')^{\frac{1}{2}} - (aa')^{\frac{1}{2}}] \geq 6\sqrt[3]{9V^{\frac{1}{2}}} + [aa' - bb']^2 + [bb' - cc']^2 + [cc' - aa']^2,\]

with equalities hold if ABCD is regular.

Thus, inequality (26) and (27) may be considered as the Finsler-Hadwiger type inequality of tetrahedron. We denote by \(R\) and \(r\) the circumradius and inradius of tetrahedron ABCD. We find the following important result from [12] (Taking \(n = 3\)):

\[(28)\] \[R \geq 3r,\]

with equality holding if and only if tetrahedron ABCD is regular.

Obvious, inequality (28) is a generalization to \(E^3\) of the Euler inequality \(R \geq 2r\) in the 2-dimensional plane triangle; that is, inequality (28) is the Euler inequality in \(E^3\). We shall give a strengthening of inequality (28) by using the result of this paper.

**Theorem 4.2.** For a tetrahedron ABCD, we have

\[(29)\] \[R^2 \geq 9r^2 + [(aa')^{\frac{1}{2}} - (bb')^{\frac{1}{2}}]^2 + [(bb')^{\frac{1}{2}} - (cc')^{\frac{1}{2}}]^2 + [(cc')^{\frac{1}{2}} - (aa')^{\frac{1}{2}}]^2,\]

with equality holds if ABCD is regular.
Proof. Noting the well-known results (see [7, p.553])

\begin{align}
8R^2 & \geq aa' + bb' + cc', \\
V & \geq 8\sqrt{3}r^3,
\end{align}

with equalities holding if and only if tetrahedron $ABCD$ is regular.

From inequalities (27), (31) and (32), we can infer (30).

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