INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 2 (2013), No. 1, 54 - 59

# THREE PROOFS TO AN INTERESTING PROPERTY OF CYCLIC QUADRILATERALS

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**Abstract.** The main purpose of the paper is to present three different proofs to an interesting property of cyclic quadrilaterals contained in the Theorem in Section 2.

## 1. INTRODUCTION

There are many geometric properties involving cyclic quadrilaterals (we mention the references [1]-[5]). In this note we discuss a property which appears as "folklore" and we present three different ways to prove it. This property is contained in the statement of Theorem in Section 2, but it was proposed as a problem in a Saudi Arabia IMO Team Section Test in 2012 [6]. The first proof is in the spirit of the old fashion Geometry and it involves only the ability. The second proof uses a combination between a recent result published in the journal Kvant, the Newton-Gauss line applied in a non-standard way, and the Pappus Theorem. The last one is computational and it uses the Ceva Theorem.

## 2. MAIN RESULTS

We will present three different proofs to the following result involving cyclic quadrilaterals.

**Theorem.** *In a cyclic quadrilateral $ABCD$, diagonals $AC$ and $BD$ intersect at point $P$. Let $E$ and $F$ be the respective feet of the perpendiculars from $P$ to lines $AB$ and $CD$. Segments $BF$ and $CE$ meet at $Q$. Prove that lines $PQ$ and $EF$ are perpendicular to each other.*

**Proof 1** (*Zuming Feng, Philip Exeter Academy, USA*). Point $H$ lies on $EF$ such that $PH \perp EF$, and point $R_E$ lies on $PH$ such that $ER_E \perp BF$.

**Keywords and phrases:** cyclic quadrilateral; Newton-Gauss line of a quadrilateral; Pappus theorem
**(2010)Mathematics Subject Classification:** 97G40, 51M04.
Received: 21.12.2012 In revised form: 20.01.2013 Accepted: 15.02.2013

Consider the point $X$ defined by $\{X\} = ER_E \cap BF$ (see Figure 1).

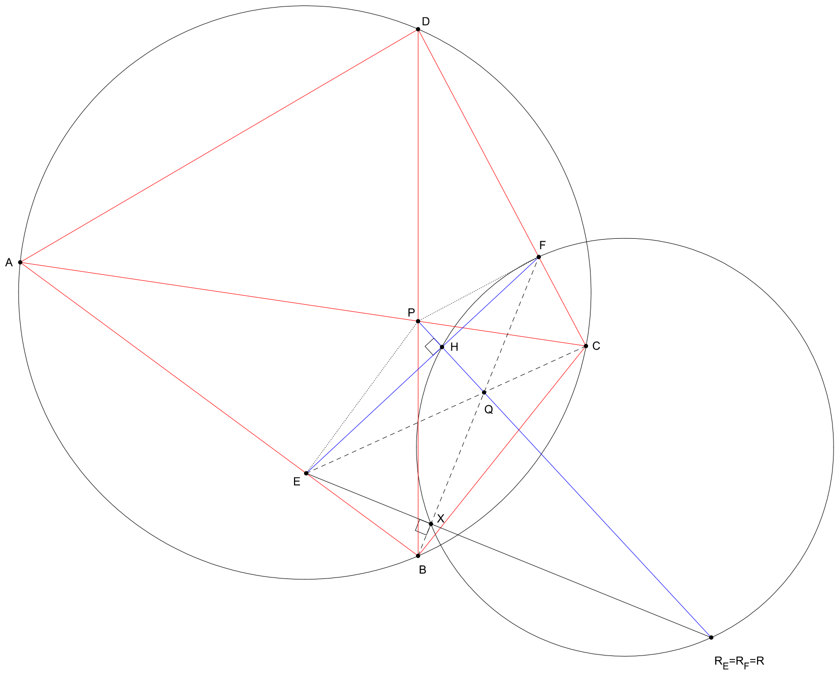


Figure 1

Because $\widehat{HR_EX} = \widehat{HFX} = 90°$, the quadrilateral $XR_EFH$ is cyclic. It follows

$$\widehat{PR_EE} = \widehat{PR_EX} = \widehat{PR_EX} = \widehat{HFX} = \widehat{EFB}. \tag{1}$$

Also note that

$$\widehat{R_EEP} = 90° - \widehat{BER_E} = 90° - \widehat{BEX} = \widehat{XBE} = \widehat{FBE}. \tag{2}$$

By (1) and (2), we know that $PR_EE$ and $EFB$ are similar, implying that

$$\frac{PR_E}{EF} = \frac{EP}{BE}. \tag{3}$$

Define $R_F$ as point on $PH$ such that $FR_F \perp CE$. In exactly the same way, we can show that

$$\frac{PR_F}{EF} = \frac{FP}{CF}. \tag{4}$$

Because $ABCD$ is cyclic, triangle $ABP$ and $DCP$ are similar, and $E$ and $F$ are corresponding points under this similarity. In particular,

$$\frac{FP}{CF} = \frac{EP}{BE}. \tag{5}$$

By (3), (4), (5), we have

$$\frac{PR_E}{EF} = \frac{PR_F}{EF},$$

hence $R_E = R_F = R$. Now we see that $Q$ is the orthocenter of triangle $EFR$, in particular, $RQ \perp EF$. By definition of $R$, we have $RP \perp EF$ so $PQ \perp EF$. □

**Proof 2.** We will use the following auxiliary result.

**Lemma** (Kvant, 2007). *Let $M$, $N$ be the midpoints of $BC$ and $AD$, respectively. We have $MN \perp EF$, and quadrilateral $MFNE$ is a kite, that is $MN$ passes through the midpoint of $EF$.*

**Proof of Lemma.** Let $K$, $L$ be the midpoints of $AP$ and $DP$. We can prove immediately that triangle $EKN$ and $FLN$ are congruent, hence $NE \equiv NF$ (see Figure 2).

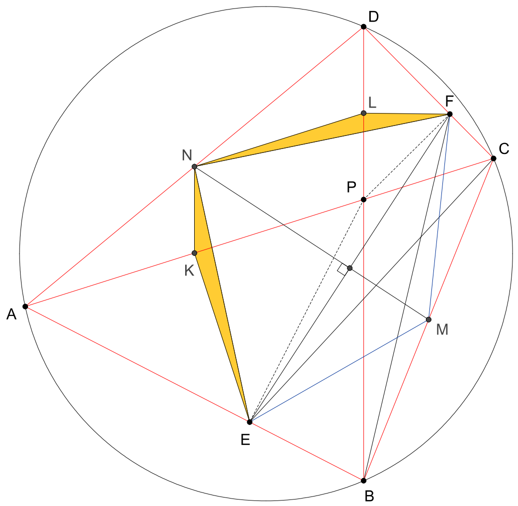


Figure 2

Similarly, we obtain $ME = MF$. From the congruence of triangles $MEN$ and $MFN$, it follows that $E$ and $F$ are symmetric with respect to the line $MN$ and we are done. $\square$

**Remark 1.** The converse of the property in the above Lemma is also true in the following form. With the notations above, if for a convex quadrilateral $ABCD$ we have $MN \perp EF$, then $ABCD$ is cyclic or a trapezoid. Indeed, introducing the notations $\widehat{APB} = \widehat{CPD} = \pi - \alpha, \widehat{PAB} = x, \widehat{PBA} = \alpha - x, \widehat{PDC} = y, \widehat{PCD} = \alpha - y$ (see Figure 3),

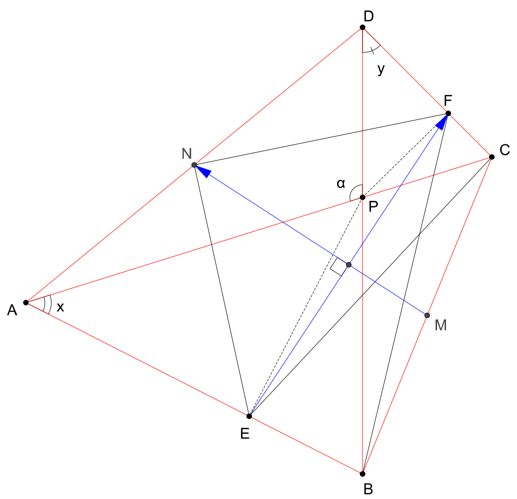


Figure 3

we have

$$\overrightarrow{EF}\cdot\overrightarrow{MN} = (\overrightarrow{PF}-\overrightarrow{PE})\cdot(\overrightarrow{PN}-\overrightarrow{PM}) = \frac{1}{2}(\overrightarrow{PF}-\overrightarrow{PE})\cdot(\overrightarrow{PB}+\overrightarrow{PC}-\overrightarrow{PA}-\overrightarrow{PD}) =$$

$$= \frac{1}{2}(\overrightarrow{PF}-\overrightarrow{PE})\cdot(\overrightarrow{AB}+\overrightarrow{DC}) = \frac{1}{2}(\overrightarrow{PF}\cdot\overrightarrow{AB}-\overrightarrow{PE}\cdot\overrightarrow{DC}).$$

Therefore, $\overrightarrow{EF}\cdot\overrightarrow{MN}=0$ if and only if $\overrightarrow{PF}\cdot\overrightarrow{AB}=\overrightarrow{PE}\cdot\overrightarrow{DC}$. But, clearly we have $(\widehat{\overrightarrow{PF},\overrightarrow{AB}}) = (\widehat{\overrightarrow{PE},\overrightarrow{DC}})$, hence we obtain

$$PF\cdot AB = PE\cdot CD.$$

The last relation is equivalent to

$$\frac{2\sigma[CPD]}{CD}\cdot AB = \frac{2\sigma[APB]}{AB}\cdot CD,$$

and we get

$$\left(\frac{AB}{CD}\right)^2 = \frac{\sigma[APB]}{\sigma[CPD]} = \frac{AP\cdot PB}{CP\cdot PD}.$$

Using the Law of Sines in triangles $APB$ and $CPD$, the last relation is equivalent to

$$\frac{\cos\alpha}{\sin x\sin(\alpha-x)} = \frac{\cos\alpha}{\sin y\sin(\alpha-y)},$$

hence

$$\sin x\sin(\alpha-x) = \sin y\sin(\alpha-y).$$

From the last relation we get $\cos(2x-\alpha)-\cos\alpha = \cos(2y-\alpha)-\cos\alpha$, that is $\cos(2x-\alpha)=\cos(2y-\alpha)$.It follows

$$-2\sin(x+y-\alpha)\sin(x-y)=0,$$

implying $x=y$ or $x=\alpha-y$. In the first case, we obtain that $ABCD$ is cyclic. The equality $x=\alpha-y$ means that the quadrilateral $ABCD$ is trapezoid.

Now, in order to prove the statement in the theorem it is enough to show that $PQ$ is parallel to $MN$.

The first step is to consider the point $\{K\}=AB\cap CD$. If $AB\|CD$ the property is clear.

According to the Newton-Gauss line applied for the points $A,B,C,D$ with diagonals $AD$ and $BC$, it follows that the midpoint $U$ of the segment $PK$ belongs to the line $MN$. From the previous Lemma it follows that the midpoint $G$ of $EF$ is also situated on the line $MN$ (see Figure 4).

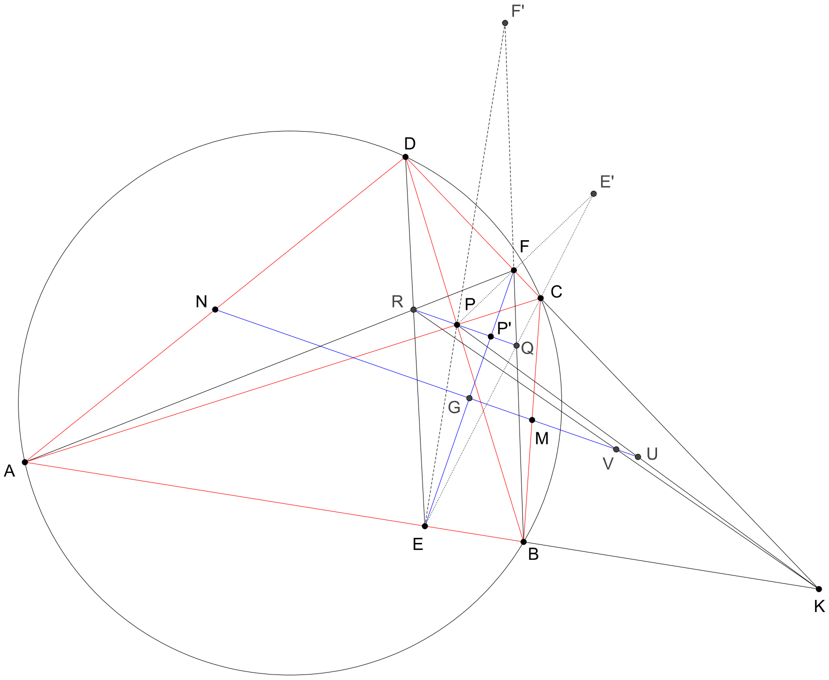


Figure 4

Using Pappus Theorem for the triples $(A, E, B)$ and $(D, F, C)$ it follows that the point $\{R\} = AF \cap DE$ is on the line $PQ$. Applying the Newton-Gauss line for the points $A, E, F, D$ with diagonals $AD$ and $EF$ we get that the midpoint $V$ of segment $RK$ is on $NG$, hence on $NM$. It follows that the line $MN$ is exactly the line $UV$.

Consider the homothecy $H_{K,1/2}$, we have $P \to U$ and $R \to V$, hence the line $PQ$ is transformed in the line $MN$, that is $PQ \| MN$. $\square$

**Proof 3** (*Malik Talbi, King Saud University, Riyadh, Saudi Arabia*)**.** If $AB \| CD$ then $P = Q$ and lies on $EF$. If not, the problem is equivalent to prove that $BF$, $CE$ and the altitude of $PEF$ at $P$ are concurrent. Since $ABCD$ is cyclic $\widehat{PBE} = \widehat{FCP}$ and the two right triangles $EBP$ and $FPC$ are similar, we use the notations in Figure 5.

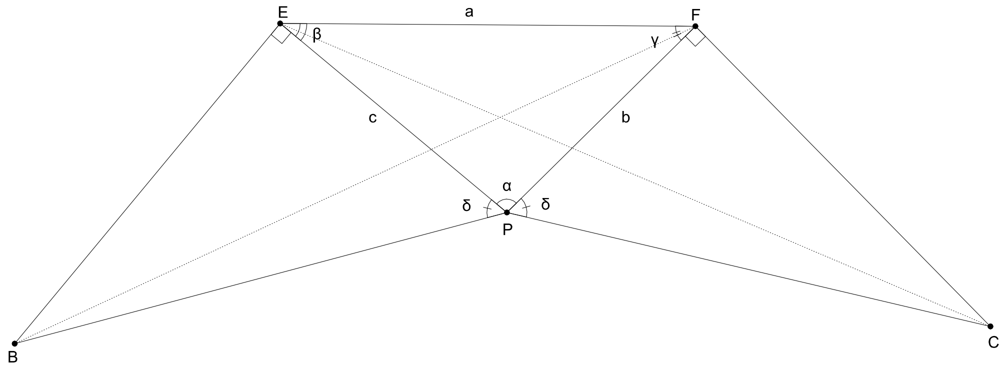


Figure 5

**Remark 2.** If $BF$, $EC$ intersect outside $EPF$, the same argument occurs with some modifications.

Let $F'$ be the intersection of $BF$ with line $EP$, $E'$ be the intersection of $CE$ with line $FP$. We have from the area of triangle $CEP$

$$\frac{1}{2}PC \cdot PE \sin(\alpha+\delta) = \frac{1}{2}PC \cdot PE' \sin\delta + \frac{1}{2}PE \cdot PE' \sin\alpha.$$

Then

$$PE' = \frac{PC \cdot PE \sin(\alpha+\delta)}{PC\sin\delta + PE\sin\alpha} = \frac{bc\sin(\alpha+\delta)}{b\sin\delta + c\sin\alpha\cos\delta}$$

$$E'F = b - PE' = b\frac{b\sin\delta - c\cos\alpha\sin\delta}{b\sin\delta + c\sin\alpha\cos\delta}.$$

Therefore

$$\frac{PE'}{E'F} = \frac{c\sin(\alpha+\delta)}{b\sin\delta - c\cos\alpha\sin\delta}.$$

In a similar way

$$\frac{PF'}{F'E} = \frac{b\sin(\alpha+\delta)}{c\sin\delta - b\cos\alpha\sin\delta}.$$

Let $P'$ be the foot of the altitude of $EFP$ at $P$. We have

$$\frac{FP'}{P'E} = \frac{b\cos\gamma}{c\cos\beta}.$$

Hence

$$\frac{PE'}{E'F} \cdot \frac{FP'}{P'E} \cdot \frac{EF'}{F'P} = \frac{\cos\gamma(c - b\cos\alpha)}{\cos\beta(b - c\cos\alpha)}$$

$$= \frac{\cos\gamma(\sin\gamma - \sin\beta\cos\alpha)}{\cos\beta(\sin\beta - \sin\gamma\cos\alpha)} = \frac{\cos\gamma\cos\beta\sin\alpha}{\cos\beta\cos\gamma\sin\alpha} = 1.$$

We deduce from Ceva theorem that $PP'$, $EE'$, $FF'$ are concurrent. □

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