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ATIYAH'S CONJECTURE REVISITED

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Abstract. The Atiyah's conjecture is a problem in elementary space geometry that has arisen from M.V.Berry and J.M. Robbins investigations on the spin-statistics theorem in quantum mechanics. Our aim is to state and prove a variation of Atiyah conjecture in the Euclidean plane which preserves the spirit of the space Atiyah's conjecture.

1. Introduction

Let us remember the statement of the conjecture as it appears in [4]. *Consider* n *distinct ordered points* $X_i \in \mathbb{R}^3, i = \overline{1,n}$. *For each pair* $i \neq j$ *define the unit vector* $P_{ij} = \frac{X_i - X_j}{|X_i - X_j|}$. *The stereographic projection corresponding to* $N(0,0,1)$ *and the plane* xOy *allows us to define* $T_{ij} = NP_{ij} \cap (xOy)$. *Let* $z_{ij} \in \mathbb{C}$ *be the affix of* T_{ij}. *Set* P_i *to be the polynomial in* z *with roots* $z_{ij}, i \neq j$, *that is*

$$P_i = \prod_{j \neq i} (z - z_{ij}).$$

Atiyah assertion is that for any $X_1, X_2, ..., X_n \in \mathbb{R}^3$ *the polynomials* P_1, P_2, ..., P_n *are linearly independent over* $\mathbb{C}$.

The Atiyah's conjecture connection with physics via geometric energies is explained in [1], [3] and [6]. In [2] one can see the proof of the conjecture in the case $n = 3$. Some interesting particular cases are investigated in [4] and a complete solution in the case $n = 4$ one can find in [5].

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2. A variation of the Atiyah's conjecture

The original Atiah's conjecture in the case $n = 3$ leads to a plane configuration obtained by the author starting from the space problem after the study of some transformations which leave invariant a given determinant. In this way the polynomials are created with six points lying on a circle. In order to preserve the statement seen in the introduction we propose the following variation of Atiyah's conjecture in which the six points are not necessarily lying on the same circle.

Theorem 2.1. *Given three points X_1, X_2, X_3 in the plane let us denote by $P_{ij} = \frac{X_i - X_j}{|X_i - X_j|}$ $i, j = \overline{1,3}$ the six points determined on the unit circle centered in O. Denote the coordinates of P_{ij} by (x_{ij}, y_{ij}). Consider N the point having the coordinates $(0, 1)$. The lines NP_{ij} cut the x-axis in A_{ij} with the coordinates $\frac{-x_{ij}}{y_{ij}-1}$. Consider the polynomials $P_i(x) = \left(X + \frac{x_{ij}}{y_{ij}-1}\right)\left(X + \frac{x_{ik}}{y_{ik}-1}\right)$, $i, j, k \in 1, 2, 3, i \neq j, i \neq k, j \neq k$. Then, the three polynomials are linearly independent over $\mathbb{C}$.*

Proof. Let us observe that P_{ij} and P_{ji} are antipodal, therefore

$$\begin{cases} x_{ij} = -x_{ji} \\ y_{ij} = -y_{ji} \end{cases}, i, j = \overline{1,3}, i \neq j.$$

If we denote $\begin{cases} x_{12} &= a_1, & y_{12} &= b_1 \\ x_{23} &= a_2, & y_{23} &= b_2 \\ x_{31} &= a_3, & y_{31} &= b_3 \end{cases}$ the polynomials become

$$\begin{cases} P_1(x) = \left(X + \frac{a_1}{b_1-1}\right)\left(X + \frac{a_3}{b_3+1}\right) \\ P_2(x) = \left(X + \frac{a_1}{b_1+1}\right)\left(X + \frac{a_2}{b_2-1}\right) \\ P_3(x) = \left(X + \frac{a_3}{b_3-1}\right)\left(X + \frac{a_2}{b_2+1}\right) \end{cases}$$

Since $(a_i, b_i), i = \overline{1,3}$ are the coordinates of three points which belong to a circle we have $a_i^2 + b_i^2 = 1, i = \overline{1,3}$. So, it exists $\beta_i \in [0, 2\pi), i = \overline{1,3}$ such that $a_i = \sin\beta_i$ and $b_i = \cos\beta_i, i = \overline{1,3}$. It results:

$$\begin{cases} \frac{a_i}{b_i-1} = \frac{\sin\beta_i}{\cos\beta_i-1} = -\cot\frac{\beta_i}{2} \\ \\ \frac{a_i}{b_i+1} = \frac{\sin\beta_i}{\cos\beta_i+1} = \tan\frac{\beta_i}{2}, \ i = \overline{1,3} \end{cases}$$

If we denote $\tan\frac{\beta_i}{2} = \alpha_i, \ i = \overline{1,3}$ the polynomials can be written in the form

$$P_1(x) = X^2 - \left(\frac{1}{\alpha_1} - \alpha_3\right)X - \frac{\alpha_3}{\alpha_1}$$

$$P_2(x) = X^2 - \left(\frac{1}{\alpha_2} - \alpha_1\right)X - \frac{\alpha_1}{\alpha_2}$$

$$P_3(x) = X^2 - \left(\frac{1}{\alpha_3} - \alpha_2\right)X - \frac{\alpha_2}{\alpha_3}$$

They are linearly independent if and only if the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{\alpha_1} - \alpha_3 & \frac{1}{\alpha_2} - \alpha_1 & \frac{1}{\alpha_3} - \alpha_2 \\ \frac{\alpha_3}{\alpha_1} & \frac{\alpha_1}{\alpha_2} & \frac{\alpha_2}{\alpha_3} \end{vmatrix}$$

is nonzero.
By calculating we have:

$$\Delta = -\frac{1}{\alpha_1\alpha_2\alpha_3} \cdot \frac{1}{2}[(\alpha_1\alpha_2 - \alpha_2\alpha_3)^2 + (\alpha_1\alpha_2 - \alpha_3\alpha_1)^2 + (\alpha_2\alpha_3 - \alpha_3\alpha_1)^2 +$$
$$+(\alpha_1 - \alpha_2)^2 + (\alpha_2 - \alpha_3)^2 + (\alpha_3 - \alpha_1)^2]$$

If $\Delta = 0$ we have $\alpha_1 = \alpha_2 = \alpha_3$, that is $P_1 = P_2 = P_3$. From $P_1 = P_2$ we deduce $A_{12} = A_{23}$ and $A_{13} = A_{21}$ because of $A_{12} \neq A_{21}$. In the same way from $P_1 = P_3$ it results $A_{13} = A_{32}$ and $A_{12} = A_{31}$.
It means $A_{12} = A_{23} = A_{31}$ and $A_{21} = A_{32} = A_{13}$, therefore we have both $P_{12} = P_{23} = P_{31}$ and $P_{21} = P_{32} = P_{13}$. The situation described above is happening if and only if $X_1 = X_2 = X_3$. It results that Δ can not vanish, that is the three polynomials are linearly independent over $\mathbb{C}$.

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