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INSCRIBED CIRCLE OF GENERAL SEMI-REGULAR POLYGON AND SOME OF ITS FEATURES

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Abstract.
If above each side of a regular polygon with n sides, we construct an isosceles polygon with k-1 equal sides we get an equilateral polygon with $N = (k-1)n$ equal sides and different interior angles-semi regular polygons. Some metrical features and relations relating to the inscribed circle of the general semi-regular equilateral polygon with $N = (k-1)n$ sides, and with $n, k \geq 3, k, n \in \mathbf{N}$ are dealt with in this paper. Furthermore, the paper contains a proof to the theorem on geometrical construction of the semi-regular polygon with $N = (k-1)n$ sides, given a radius of an inscribed circle.

1. Introduction

Given the set of points $A_j \in E^2, j = 1, 2, \ldots, n$ in Euclidian plane E^2, such that any three successive points do not lie on a line p and for which we have a rule: if $A_j \in p$ and $A_{j+1} \in p$ for each j point A_{j+2} does not belong to the line p.

1. Polygon P_n or closed polygonal line is the union along $A_1A_2, A_2A_3, \ldots, A_nA_{n+1}$, and write short

$$P_n = \bigcup_{j+1}^{n} A_jA_{j+1}, (n+1 \equiv 1 \mod n) \tag{1}$$

Points A_j are vertices, and lines A_jA_{j+1} are sides of polygon P_n.

2. The angles on the inside of a polygon formed by each pair of adjacent sides are angles of the polygon.

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3. If no pair of polygon's sides, apart from the vertex, has no common points, that is ,if $A_jA_{j+1} \cap A_{j+l}A_{j+l+1} = \emptyset, l \neq 1$ polygon is simple, otherwise it is complex. This paper deals with simple polygons only.

4. Simple polygons can be convex and non-convex. Polygon is convex if it all lies on the same side of any of the lines A_jA_{j+1}, otherwise it is non-convex. Polygon P_n divides plane E^2 into two disjoint subsets,U and V. Subset U is called interior, and subset V is exterior area of the polygon. Union of polygon P_n and its interior area U_n makes *polygonal area* S_n, which is:

$$S_n = P_n \cup U_n \tag{2}$$

5. Given polygon P_n with vertices $A_j, j = 1, 2, \ldots, n, (n + 1 \equiv 1 \mod n)$ lines of which A_jA_i are called polygonal diagonals if indices are not consecutive natural numbers, that is, $j \neq i$. We can draw $n - 3$ diagonals from each vertex of the polygon with n number of vertices.

6. Exterior angle of the polygon P_n with vertex A_j is the angle $\angle A_{v,j}$ with one side $A_{j+1}A_j$, and vertex A_j, and the other one is extension of the side A_jA_{j-1} through vertex A_j.

7. Sum of all exterior angles of the given polygon P_n is equal to multiplied number or product of tracing around the polygons in a certain direction and 2π, that is, the rule is

$$\sum_{j=1}^{n} (\angle A_{v,j}) = 2k\pi, k \in \mathbb{Z} \tag{3}$$

In which k is number of turning around the polygon in certain direction.

8. The interior angle of the polygon with vertex A_j is the angle $\angle A_{u,j}, j = 1, 2, \ldots, n$ for which $\angle A_{u,j} + \angle A_{v,j} = \pi$. That is the angle with one side $A_{j-1}A_j$, and the other side A_jA_{j+1}. Sum of all interior angles of the polygon is defined by equation

$$\sum_{j=1}^{n} \angle A_{u,j} = (n - 2k)\pi, n \in \mathbb{N}, k \in \mathbb{Z}. \tag{4}$$

In which k is number of turning around the polygon in certain direction.

9. A regular polygon is a polygon that is equiangular (all angles are equal in measure) and equilateral (all sides have the same length). Regular polygon with n sides of b length is marked as P_n^b. The formula for interior angles γ of the regular polygon P_n^b with n sides is $\gamma = \frac{(n-2)\pi}{n}$. A non-convex regular polygon is a regular star polygon.For more about polygons in [4,5,6].

10. Polygon that is either equiangular or equilateral is called *semi-regular polygon*. Equilateral polygon with different angles within those sides are called *equilateral semi-regular* polygons, whereas polygons that are *equiangular* and with sides different in length are called *equiangular* semi regular polygons. For more about polygons in [1,2,3].

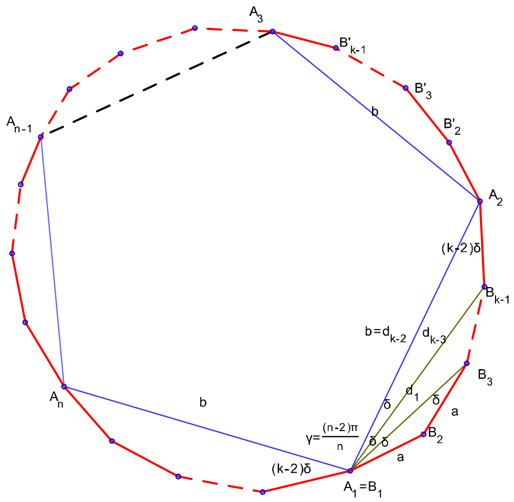

FIGURE 1. Convex semi-regular polygon P_N with $N = (k-1)n$ sides constructed above the regular polygon P_n^b

11. If we construct a polygon P_k with $(k-1)$ sides, $k \geq 3, k \in \mathbb{N}$ with vertices $B_i, i = 1, 2, \dots, k$ over each side of the convex polygon $P_n, n \geq 3, n \in \mathbb{N}$ with vertices $A_j, j = 1, 2, \dots, n$, $(n+1) \equiv 1 \mod n$, that is $A_j = B_1, A_{j+1} = B_k$, we get new polygon with $N = (k-1)n$ sides, (Figure 1) marked as P_N.

Here are the most important elements and terms related to constructed polygons:

(1) Polygon P_k with vertices $B_1B_2 \dots B_{k-1}B_k$, $B_1 = A_j$, $B_k = A_{j+1}$ constructed over each side A_jA_{j+1}, $j = 1, 2, \dots, n$ of polygon P_n with which it has one side in common is called edge polygon for polygon P_n.

(2) $A_jB_2, B_2B_3, \dots, B_{k-1}A_{j+1}$, $j = 1, 2, \dots, n$ are the sides polygon P_k.

(3) $A_jB_2A_jB_3, \dots, A_jB_{k-1}$ are diagonals $d_i, i = 1, 2, \dots, k-2$, of the polygon P_k^a drawn from the top A_j and that implies

$$d_{k-2} = A_jA_{j+1} = b.$$

(4) Angles $\angle B_{u,i}$ are interior angles of vertices $B_{u,i}$ of the polygon P_N and are denoted as β_i. Interior angle $\angle A_{u,j}$ of the polygon of the vertices A_j are denoted as α_j.

(5) Polygon P_k of the side a constructed over the side b of the polygon P_n is isosceles, with $(k-1)$ equal sides, is denoted as P_k^a.

(6) $\delta = \angle(d_i, d_{i+1})$, $i = 1, 2, \dots, k-2$ denotes the angle between its two consecutive diagonals drawn from the vertices $A_j, j = 1, 2, \dots, n$ for which it is true

$$\delta = \angle(a, d_1) = \angle(d_id_{i+1}), i = 1, 2, \dots, k-3, d_{k-2} = b \tag{5}$$

(7) If the isosceles polygon P_k^a is constructed over each side of the b regular polygon P_n^b with n sides, then the constructed polygon with $N = (k-1)n$ of equal sides is called equilateral *semi-regular polygon* which is denoted as P_N^a.

12. We analyzed here some metric characteristics of the general equilateral semi-regular polygons, if side a is given, and angle is $\delta = \angle(d_i, d_{i+1})$, $i = 1, 2, \ldots, (k-2)$, in between the consecutive diagonals of the polygon P_k^a drawn from the vertex P_k^a of the regular polygon P_n^b. Such semi-regular polygon with $N = (k-1)n$ sides of a length and angle δ defined in (5) we denote as $P_N^{a,\delta}$.

13. Regular polygon P_n^b polygon is called corresponding regular polygon of the semi-regular polygon $P_N^{a,\delta}$.

14. Interior angles of the semi-regular equilateral polygon is divided into two groups
- angles at vertices B_i, $i = 2, \ldots, k-1$ we denote as β,
- angles at vertices A_j, $j = 1, 2, \ldots, n$ we denote as α.

15. K_N stands for the sum of the interior angles of the semi-regular polygon $P_N^{a,\delta}$.

16. $S_{A_j}^{\gamma}$ stands for the sum of diagonals comprised by angle γ and drawn from the vertex A_j, and with $\varepsilon_{A_j}^{\gamma}$ we denote the angle between the diagonals drawn from vertex A_j comprised by angle γ.

17. We denote the radius of the inscribed circle of the semi-regular polygon $P_N^{a,\delta}$, with r_N.

2. MAIN RESULT

Let on each side of the regular polygon P_n^b, be constructed polygon P_k^a, with $(k-1)$ equal sides, and let $d_l = A_jB_i, l = 1, 2, \ldots, k-2, d_{k-2} = A_jA_{j+1} = b, j = 1, 2, \ldots, n; i = 3, 4, \ldots, k; B_k = A_{j+1}$ diagonals drawn from the vertex A_j,$A_jA_{j+1} = b$ to the vertices B_i of the polygon P_k^a. The following lemma is valid for interior angles at vertices $B_i, B_k = A_{j+1}$ of triangle $\triangle A_jB_{i-1}B_i$ determined by diagonals d_i. For more about in [7].

Lemma 2.1. *Ratio of values of interior angles* $\triangle A_jB_{i-1}B_i; i = 3, 4, \ldots, k$ *at vertex* B_i *of the base* $A_jB_i = d_{i-2}$ *from the given angle* $A_jB_i = d_{i-2}$ δ *is defined by relation* $\angle B_i = (i-2)\delta$.

Proof. The proof is done by induction on i, $(i \geq 3), i \in \mathbb{N}$. Let us check this assertion for $i = 4$ because for $i = 3$ the claim is obvious because the triangle erected on the sides of the regular polygon is isosceles and angles at the base b are equal as angle δ. If $i = 4$ and isosceles rectangle is constructed on side b of the regular polygon P_n^b (Figure 2) with vertices $A_1B_2B_3B_4$, and $B_4 = A_2$ where $A_1A_2 = b$ side of the regular polygon.

Diagonals constructed from the vertex A_1 divide polygon $A_1B_2B_3B_4$, into triangles $\triangle A_1B_2B_3$ and $\triangle A_1B_3B_4$. According to the definition of the angle δ we have:

$$\angle B_2A_1B_3 = \angle B_2B_3A_1 = \angle B_3A_1B_4 = \delta$$

Intersection of the centerline of the triangle's base $\triangle A_1B_2B_3$ i $A_1B_4 = b$ is point S_1. Since $A_1B_2 = B_2B_3 = a$, a and construction of the point S_1 leads to conclusion that $\square A_1S_1B_2B_3$ is a rhombus with side a.

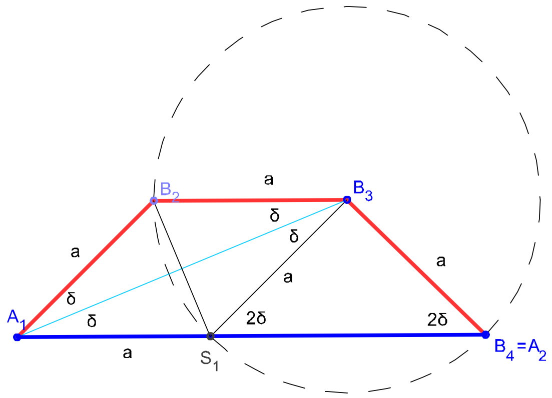

FIGURE 2. Rectangle $A_1B_2B_3B_4$

Since $B_3S_1 = B_3B_4 = a$ a triangle $\triangle B_3S_1B_4$ is isosceles, and its interior angle at vertex S_1 is exterior angle of the triangle $\triangle A_1B_3S_1$, thus $\angle S_1 = 2\delta$, as well as $\angle B_4 = 2\delta$. Let us presume that the claim is valid for an arbitrary integer $(p-1)$, $(p \geq 4)$, $p \in \mathbb{N}$, that is $i = (p-1)$ interior angle of the triangle $\triangle A_jB_{p-2}B_{p-1}$ at the vertex B_{p-1} has value $\angle B_{p-1} = (p-3)\delta$.

Let us show now that this ascertain is true for integer p, that is for $i = p$. Also, interior angle of the triangle $\triangle A_jB_{p-1}B_p$ at vertex B_p has value $\angle B_p = (p-2)\delta$. Let us note $\square A_jB_{p-2}B_{p-1}B_p$ which is split into triangles $\triangle A_jB_{p-2}B_{p-1}$ and $\triangle A_jB_{p-1}B_p$ by diagonal d_{p-3}, and that $\angle B_{u,p-1} = (p-3)\delta$ according to presumption (Figure 3).

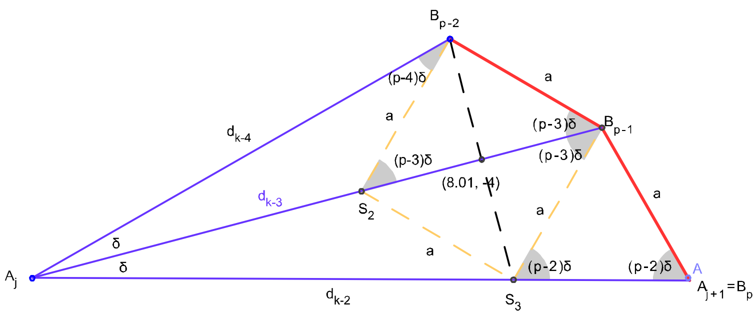

FIGURE 3. Rectangle $A_jB_{p-2}B_{p-1}B_p$

Since interior angles of triangles are congruent at vertex A_j, by definition of angle δ, and $B_{p-2}B_{p-1} = B_{p-1}B_p = a$, it is easily proven that there is point S such that triangle $\triangle SB_{p-1}B_p$ is isosceles triangle (Figure 3.), and rectangle $\square A_jB_{p-2}B_{p-1}S$ is rectangle with perpendicular diagonals.

Congruence of triangles $\triangle A_jB_{p-2}B_{p-1} \simeq \triangle A_jB_{p-1}S$ leads us to conclusion that

$$\angle A_jB_{p-1}S = (p-3)\delta.$$

Angle at vertex S is the exterior angle of triangle $\triangle A_j B_{p-2} B_{p-1}$. And thus we have

$$\angle S = \delta + (p-3)\delta = (p-2)\delta.$$

Since triangle $SB_{p-1}B_p$ is isosceles, $\angle B_p = (p-2)\delta$ which we were supposed to prove. So, for each $i \in \mathbb{N}, i \geq 3$ interior angle of triangle $\triangle A_j B_{i-1} B_i$ at vertex B_i is $\angle B_i = (i-2)\delta$. $\square$

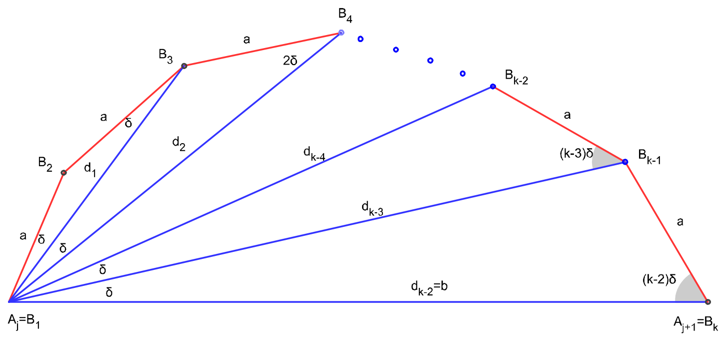

FIGURE 4. Isosceles polygon P_k^a constructed on side b of the regular polygon P_n^b

Lemma 2.2. *Semi regular equilateral polygon $P_{(k-1)n}^{a,\delta}$ with given side a and angle δ defined with (5), has n interior angles equal to an that angle*

$$\alpha = \frac{(n-2)\pi}{n} + 2(k-2)\delta \tag{6}$$

and $(k-2)n$ interior angles equal to an that angle

$$\beta = \pi - 2\delta, \delta > 0, k \geq 3, n \geq 3, k, n \in \mathbb{N} \tag{7}$$

Proof. Using figure 4 and results of lemma 2.1 it is easily proven that polygon P_k^a constructed on side b of the regular polygon P_n^b has $(k-2)$ interior angles with value $\pi - 2\delta$, and which are at the same time interior angles of the semi-regular polygon $P_N^a, N = (k-1)n$. So indeed, for $k = 3$ the constructed polygon P_k is isosceles triangle with interior angle at vertex $B_2 = \pi - 2\delta$, and for $k = 4$ constructed polygon is isosceles rectangle (Figure 2). That rectangle is drawn by diagonal d_1 from vertex $A_j, j = 1, 2, \dots, n$, $B_1 \equiv A_1, B_4 \equiv A_{j+1}$ and $A_j A_{j+1} = b$ split into triangles $A_j B_2 B_3$ and $A_j B_3 A_{j+1}$ with interior angles at vertices $\angle B_2 = \angle B_3 = \pi - 2\delta$. Similarly it is proven that for every rectangle $A_j B_{i-2} B_{i-1} B_i$, $i = 4, 5, \dots, k$; $(B_1 = A_j$, $B_k = A_{j+1}$, $A_j A_{j+1} = b)$ and the value of its vertex B_{i-1},

$$\angle B_{i-1} = (i-3)\delta + \pi - [(i-2)\delta + \delta] = \pi - 2\delta.$$

So, in every isosceles polygon P_k^a there $k-2$ interior angles with measure $\pi - 2\delta$ (Figure 4.). $\square$

Since isosceles polygon P_k^a, is constructed on each side of regular polygon P_n^b, it follows that equilateral semi-regular polygon P_N^a has total of $(k-2)n$ angles, which we were supposed to prove.

When interior angle of the semi-regular equilateral polygon at vertex A_j, $j = 1, 2, \ldots, n$ is equal to sum of interior angle of the regular polygon P_n^b and double value of the interior angle of the polygon P_k^a at vertex B_k, (Lemma 2.1) is valid

$$\angle A_{u,j} = \alpha = \frac{(n-2)\pi}{n} + 2(k-2)\delta$$

which we were supposed to prove.

Condition of convexity of the semi-regular equilateral polygon $P_N^{a,\delta}$ and the values of its angle δ is given in the theorem.

Theorem 2.1. *Equilateral semi-regular polygon $P_N^{a,\delta}$,$N = (k-1)n$ is convex if the following is true for the angle δ*

$$\delta \in \left(0; \frac{\pi}{(k-2)n}\right) \quad k, n \in \mathbb{N}, n, k \geq 3. \tag{8}$$

Proof. Let us write values of the interior angles of the semi-regular polygon $P_N^{a,\delta}$ defined by relations(6),(7) in the form of linear functions

$$f(\delta) = \frac{(n-2)\pi}{n} + 2(k-2)\delta, g(\delta) = \pi - 2\delta, \dot{k}, n \in \mathbb{N}, \dot{k}, n \geq 3. \tag{9}$$

Since the polygon is convex if all its interior angles are smaller than π, to prove the theorem it is enough to show that for $\forall\delta \in \left(0; \frac{\pi}{(k-2)n}\right)$ all interior angles of the semi-regular polygon $P_N^{a,\delta}$ are smaller than π.

Indeed, from this relation $\beta = g(\delta) = \pi - 2\delta$ follows that $\beta = 0$ for $\delta = \frac{\pi}{2}$, (Figure 5). On the basis of this and demands $\beta > 0$ and $\delta > 0$, we find that $\beta \in (0, \pi)$ and $0 < \delta < \frac{\pi}{2}$, and thus we have

$$\delta \in \left(0; \frac{\pi}{(k-2)n}\right), k \geq 3.$$

It is similar for interior angles equal to angle α, (Figure 5). If we multiply the inequality $0 < \delta < \frac{\pi}{(k-2)n}$ with $2(k-2)$, and $\frac{(n-2)\pi}{n}$ then add to the left and right side , we get the inequality

$$\begin{aligned} \frac{(n-2)\pi}{n} &< \frac{(n-2)\pi}{n} + 2(k-2)\delta < \frac{2\pi}{n} + \frac{(n-2)\pi}{n} \Leftrightarrow \\ \frac{(n-2)\pi}{n} &< \alpha < \pi, \Rightarrow \alpha \in \left(\frac{(n-2)\pi}{n}, \pi\right) \end{aligned}$$

for $\delta \in \left(0; \frac{\pi}{(k-2)n}\right), k \geq 3.$

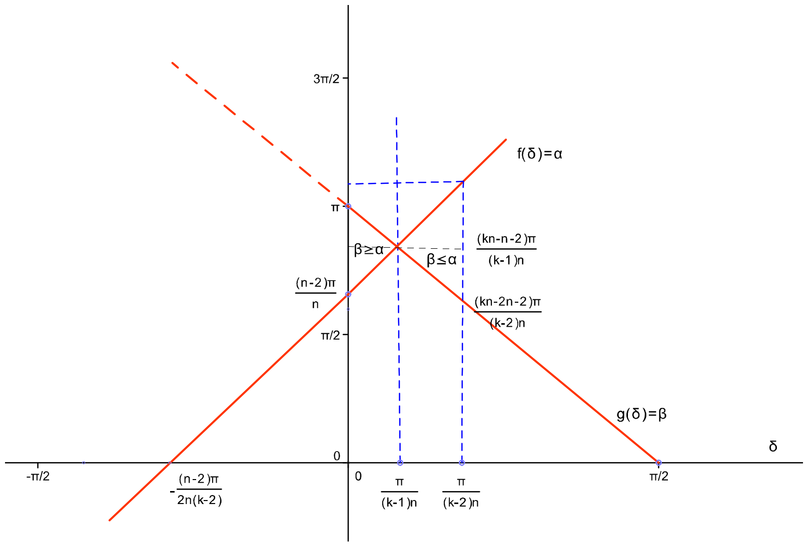

FIGURE 5. Semi-regular polygon and convexity

So, for every $\delta \in \left(0, \frac{\pi}{(k-2)n}\right)$ interior angles of the semi-regular polygon $P_N^{a,\delta}$ are smaller than π. That is, semi-regular equilateral polygon $P_N^{a,\delta}$ is convex for $\delta \in \left(0; \frac{\pi}{(k-2)n}\right)$. □

Values of the interior angles of the convex semi-regular equilateral polygon $P_N^{a,\delta}$ depend on the interior angle of the corresponding regular polygon $\gamma = \frac{(n-2)\pi}{n}$ as well as the angle δ. Which means that the following theorem is true:

Corrolary 2.1. *Convex semi-regular equilateral polygon $P_N^{a,\delta}$ is regular for $\delta = \frac{\pi}{(k-1)n}; k, n \in \mathbb{N}, n, k \geq 3, \delta > 0$ and the values of its interior angles are given in the relation*

$$\alpha = \beta = \frac{(nk - n - 2)\pi}{n(k-1)}. \tag{10}$$

Proof. According to the definition of the regular polygon, its every angle has to be equal, thus from $\alpha = \beta$ and the relation (6),(7) we have the equation

$$\frac{(n-2)\pi}{n} + 2(k-2)\delta = \pi - 2\delta$$

out of which we find out that the sought value of the angle is $\delta = \frac{\pi}{(k-1)n}$ for which the semi-regular equilateral polygon $P_N^{a,\delta}$ is regular. On this basis we find that the value of the interior angles is

$$\alpha = \beta = \frac{(nk - n - 2)\pi}{n(k-1)}.$$

□

The text further continues with the presentation of some of the results regarding the inscribed circle of the semi-regular polygon and the geometrical construction of a convex semi-regular polygon with a given radius of the inscribed circle.

Theorem 2.2. *Out of all convex equilateral semi-regular polygons with $P_N^{a,\delta}$, $N=(k-1)n$ sides constructed above the regular polygon P_n^b with n sides, a circle may be inscribed only if $k=3, \forall n \geq 3, n \in \mathbb{N}$.*

Proof. The proof is given through two stages. Firstly, let us prove that a circle may be inscribed for $P_{(k-1)n}^{a,\delta}$ for $k=3$ while it may not be possible for $P_{(k-1)n}^{a,\delta}$ with $k>3, n\geq 3, n\in\mathbb{N}$, to have an inscribed circle.

1. Let us presume that a semi-regular polygon $P_{(k-1)n}^{a,\delta}$ if $k=3, n\geq 3, n\in \mathbb{N}$ has an inscribed circle $\mathcal{C}(O,r)$ (Figure.6). Let us prove that each side of the semi-regular polygon $P_{2n}^{a,\delta}$, optional sides a and angle $\delta=\angle(a,b)$ to which it is convex, and b side of the regular polygon above which it is constructed are all tangent to such a circle. Let $A_1B_1A_2B_2\ldots A_nB_n$ be vertices, and $\angle A_i=\alpha=\frac{(n-2)\pi}{n}+2\delta, i=1,2,\ldots,n$ interior angles to vertices A_i, and $\angle B_i=\beta=\pi-2\delta, i=1,2,\ldots,n$ interior angles to vertices B_i of semi-regular polygon $P_{2n}^{a,\delta}$. Let us mark the center of inscribed circle $\mathcal{C}(O,r)$ with the mark O. If each vertex of the semi-regular polygon $P_{2n}^{a,\delta}$ is joined with the center of the inscribed circle O there can be observed the following triangles:

$$\triangle A_1OB_1, \triangle B_1OA_2, \ldots, \triangle A_nOB_n, \triangle B_nOA_1$$

for which the following is applicable:

a) $A_1B_1=A_2B_2=\cdots=A_nB_n=B_nA_1=a$ side of the semi-regular polygon,

b) $\angle OA_1B_1=\angle OA_2B_2=\cdots=\angle OA_nB_n=\frac{\alpha}{2}$

c) $\angle OB_1A_2=\angle OB_2A_3=\cdots=\angle OB_nA_1=\frac{\beta}{2}$.

Thus, we may conclude that they are mutually congruent. Let us observe one of those triangles, e.g. $\triangle A_1OB_1$. Let us mark its height from the point O to side $a=A_1B_1$ with h_1 while observing the said height to be equal to the radius of the inscribed circle $h_1=r$.

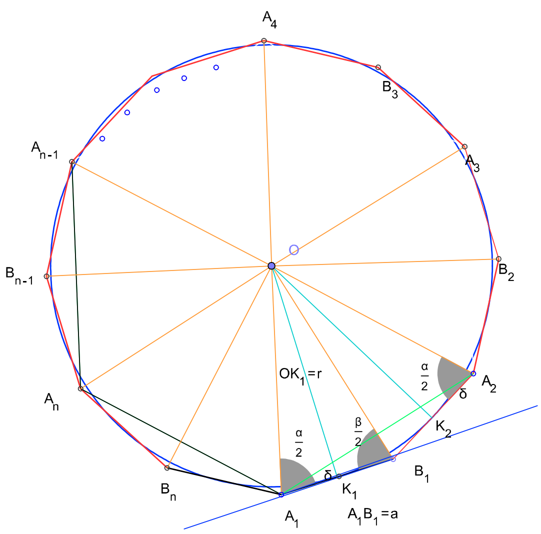

FIGURE 6. The Radius of the Inscribed Circle - Apothem

The congruency of the said triangles implies the congruency in their surfaces. If we take that $P_i = \frac{a\dot{h}_i}{2}$ is surface of triangles $\triangle A_iOB_i, i = 1, 2, \ldots, n$ that is, of triangle $\triangle B_iOA_{i+1}, i = 1, 2, \ldots, n$ i $n + 1 \equiv 1 \mod n$ with h_i being the height from vertex O to side a, then, from the equality in their surfaces, and after shortening the equation it follows that

$$h_1 = h_2 = \cdots = h_n = r \tag{11}$$

From this equation we may conclude that circle $\mathcal{C}(O, r)$ is tangent to each side of the semi-regular polygon $P_{2n}^{a,\delta}$ i.e. it is inscribed to that semi-regular polygon (Figure 6).

2. Given that for $k > 3, k \in \mathbb{N}, n \geq 3, n \in \mathbb{N}$ for semi-regular polygon $P_{(k-1)n}^{a,\delta}$ constructed above regular polygon P_n^b with n sides, there is circle $\mathcal{C}(O, r)$ inscribed with its center at point O and with radius r. By the definition of the construction of a semi-regular polygon there is an isosceles polygon P_k^a of side a constructed above each side b of regular polygon P_n^b. If the vertices of the semi-regular polygon side a become joined with point O the polygon shall become divided into two classes of mutually congruent triangles in reference to the interior angles along the base side equal to side a.

To one class of triangles belong the following:

$$\triangle OA_iB_1^j, \triangle OB_{k-1}^jA_{i+1}; i, j = 1, 2, \ldots, n; n + 1 \equiv 1 \mod n$$

which have their interior angle along vertex $\angle A_i = \frac{\alpha}{2}$, and interior angle with vertex $\angle B_j = \frac{\beta}{2}$, and base A_iB_j equal to side a of the semi-regular polygon (Figure 6).

To the other class of triangles belong the isosceles triangles $\triangle OB_p^jB_{p+1}^j$, $p = 1, 2, \ldots, k-3$, which have their angles along $B_pB_{p+1} = a$ equal to $\frac{\beta}{2}$.

From the congruence of the first class triangles follows the equality in their heights h_m^a, $m = 1, 2, \ldots, 2n$ to base a, i.e. the following is applicable:

$$h_1^a = h_2^a = \cdots = h_{2n}^a = r_1$$

with r_1 being the inscribed circle radius.

Similarly, from the congruence of the second class triangles follows the equality in their heights H_t^a, $t = 1, 2, \ldots, n(k-3)$ to base a, i.e. the following is applicable:

$$H_1^a = H_2^a = \cdots = H_{n(k-3)}^a = r_2 \tag{12}$$

with r_2 being the inscribed circle radius. Since pursuant to presumption $r_1 = r_2 = r$ what would only be possible if the first class triangles were congruent to the second class triangles, and in that case this would be applicable: $\frac{\alpha}{2} = \frac{\beta}{2} \Rightarrow \alpha = \beta$, i.e. meaning that polygon $P_{(k-1)n}^{a,\delta}$ is regular, what in turn is opposite to the presumption that it is semi-regular. Therefore, circle $\mathcal{C}(O, r)$ is not an inscribed one to the semi-regular polygon since it is not tangent to each side of the polygon , i.e. it is tangent either to base a of the first class triangle or base a of the second class triangle.

Based on the presented proof it follows that there may not be inscribed a circle to a semi-regular equilateral polygon $P_{(k-1)n}^{a,\delta}$ with $k > 3$,$n \geq 3, n \in \mathbb{N}$. □

Theorem 2.3. *Radius of the inscribed circle of the equilateral semi-regular polygon* $P^{a,\delta}_{(k-1)n}$ $k, n \geq 3, n \in \mathbb{N}$, *which does not have three consecutive vertices each with its corresponding interior angle equal to angle* $\pi - 2\delta$ *is determined through a relation*

$$r_{(k-1)n} = a\frac{\cos\delta\cos(\frac{\pi}{n} - (k-2)\delta)}{\sin(\frac{\pi}{n} - (k-3)\delta)} \tag{13}$$

Proof. Let there be vertex A_1 between the verticals constructed from center O of the inscribed circle to two neighbouring sides of the semi-regular polygon $P^{a,\delta}_{(k-1)n}$, and let there be an interior angle α as defined in a relation (6) corresponding to this vertex A_1, and let there be neighbouring vertices B_1 and B_2 with their corresponding interior angles $\beta = \pi - 2\delta$.

Now, let us observe the equiangular triangles $\triangle OB_1M, \triangle OA_1M$ with $OM = r$ (Figure 7).

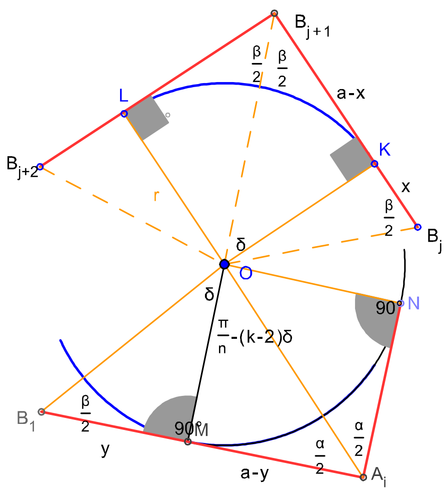

FIGURE 7. Radius of the Inscribed Circle

It is obvious from the equiangular triangle $\triangle OA_1M$ that $tg\frac{\alpha}{2} = \frac{r}{a-y}$, giving $r = (a-y)tg\frac{\alpha}{2}$.

Similarly, from equiangular triangle $\triangle OB_1M$ $tg\frac{\beta}{2} = \frac{r}{y}$, giving $y = \frac{r}{tg\frac{\beta}{2}}$. If we replace this and insert it into the first equation we get the following:

$$\frac{r}{tg\frac{\alpha}{2}} + \frac{r}{tg\frac{\beta}{2}} = a$$

and out of which we find the following:

$$r\left(\frac{1}{tg\frac{\alpha}{2}} + \frac{1}{tg\frac{\beta}{2}}\right) = a \Leftrightarrow r = \frac{a}{cot\frac{\alpha}{2} + cot\frac{\beta}{2}}$$

Since $cot\frac{\alpha}{2} = tg(\frac{\pi}{n} - (k-2)\delta)$ and $cot\frac{\beta}{2} = tg\delta$ from the previous equation, after the processing and shortening of the equation we get sought equation:

$$r_{(k-1)n} = \frac{a\cos\delta\cos(\frac{\pi}{n} - (k-2)\delta)}{\sin(\frac{\pi}{n} - (k-3)\delta)}$$

Š which needed to be proven in the first place. □

Corrolary 2.2. *The radius of the inscribed circle of the semi-regular polygon* $P_{2n}^{a,\delta}$ *is given in relation.*

$$r_{2n} = a\frac{\cos\delta\cos(\frac{\pi}{n} - \delta)}{\sin\frac{\pi}{n}} \tag{14}$$

Proof. From the relation (13) for $k = 3$ we get the sought relation. □

Theorem 2.4. *There is no convex equilateral semi-regular polygon* $P_N^{a,\delta}$ *with three consecutive vertices each with their corresponding interior angles equal to angle* $\beta = \pi - 2\delta$*, which may have a circle inscribed.*

Proof. Let us presume quite the opposite to this, i.e., that there is a semi-regular polygon $P_{n(k-1)}^{a,\delta}$ with the inscribed circle $\mathcal{C}(O,r)$ and that there is a vertex B_{j+1} with its corresponding interior angle $\beta = \pi - 2\delta$, between the verticals constructed from the center O of the inscribed circle onto the two neighboring sides. Let its neighboring vertices B_j and B_{j+2} correspond the interior angles β (Figure 7.). Then, from the equiangular triangles $\triangle OB_jK$, $\triangle OKB_{j+1}$ we find that the other relation for the radius of the inscribed circle would be as follows:

$$r_{(k-1)n} = \frac{a}{2}\cot\delta \tag{15}$$

Since the equations (13) and (15) represent the radius of inscribed circle $\mathcal{C}(O,r)$, with their processing, equaling and shortening with a, we get the following equation:

$$\frac{\cos\delta\cos(\frac{\pi}{n} - (k-2)\delta)}{\sin(\frac{\pi}{n} - (k-3)\delta)} = \frac{1}{2}\cot\delta$$

With a presumption that $\sin\delta \neq 0 \Rightarrow \delta \neq m\pi, m \in \mathbb{Z}$ and

$$\begin{aligned} \sin(\frac{\pi}{n} - (k-3)\delta) &\neq 0 \Leftrightarrow \frac{\pi}{n} - (k-3)\delta \neq l\pi, l \in \mathbb{Z} \Rightarrow \\ \delta &\neq \frac{(1-n\cdot l)\pi}{n(k-3)}, k > 3, n, k \in \mathbb{N} \end{aligned}$$

as well as with the condition of convexity of the semi-regular polygon (Theorem 2.1), this equation is then transformed into the form

$$\begin{aligned} 2\sin\delta\cos\delta\cos(\frac{\pi}{n} - (k-2)\delta) &= \sin(\frac{\pi}{n} - (k-3)\delta)\cos\delta \Leftrightarrow \\ \cos\delta &= 0 \vee 2\sin\delta\cos(\frac{\pi}{n} - (k-2)\delta) = 0. \end{aligned}$$

From the equation $cos\delta = 0$ we find that the $\delta = \frac{(2l+1)\pi}{2}$, $l \in \mathbb{Z}$ is the solution. This solution does not meet the condition of convexity of the semi-regular polygon, not even for one whole number $l \in \mathbb{Z}$.

From the second equation, being that

$$2\sin\delta\cos(\frac{\pi}{n}-(k-2)\delta)=\sin(\frac{\pi}{n}-(k-1)\delta)+\sin(\frac{\pi}{n}-(k-3)\delta)$$

we have the following

$$\sin(\frac{\pi}{n}-(k-1)\delta)=0.$$

Here we find that $\delta \neq \frac{(1-nl)\pi}{n(k-1)}, \quad l \in \mathbb{Z}, \quad ,k \geq 3, \quad n,k \in \mathbb{N}$. The only value of the angle δ which meets the condition of convexity is the one with $l=0$, and then $\delta = \frac{\pi}{n(k-1)}$. Since with such value of the angle δ polygon $P^{a,\delta}_{(k-1)n}$ is regular, it follows that there is no semi-regular polygon which has three consecutive vertices with their corresponding interior angles equal to the angle $\pi - 2\delta$ such that it can be inscribed a circle. □

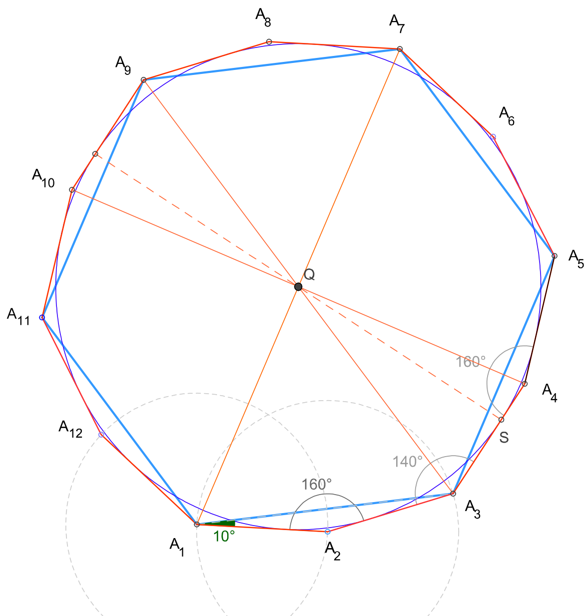

FIGURE 8. A circle may be inscribed to the equilateral semi-regular dodecagon if $k=3$,$n=6$.

Example 1. Examples of semi-regular polygon with inscribed circles;

a) For $k=3, n=6$ equilateral semi-regular dodecagon which may be inscribed a circle. (Figure 8).

b) For $k=5, n=3$ (Figure 9) and for $k=4, n=4$ (Figure 10),equilateral semi-regular dodecagon which may not be inscribed a circle. As an example, the angle value of $\delta = 10°$ has been chosen.

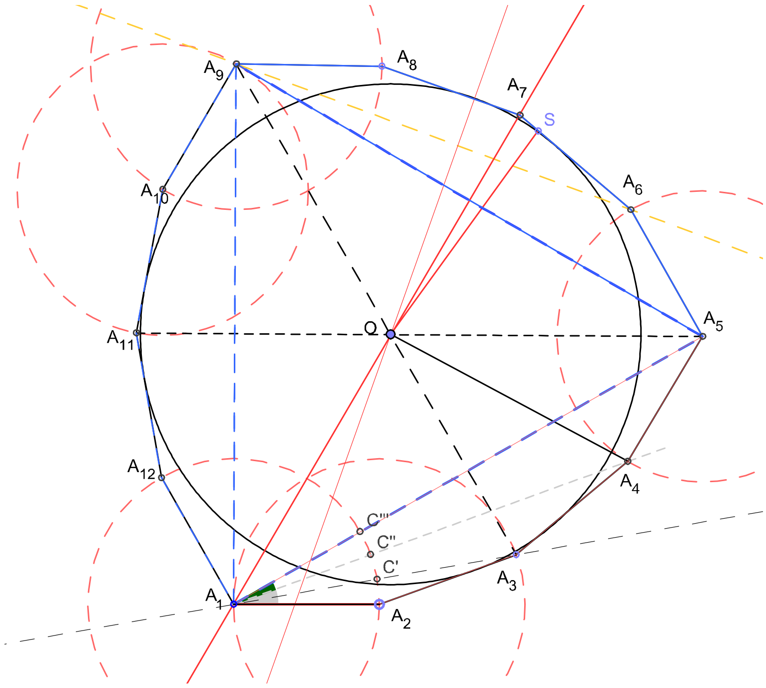

FIGURE 9. A circle may not be inscribed to the equilateral dodecagon with $k = 5, n = 3$

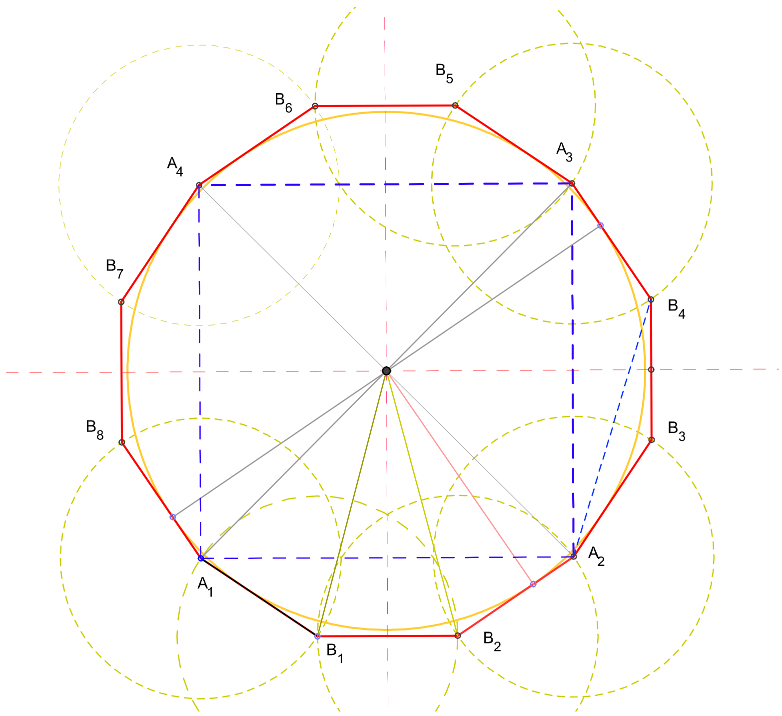

FIGURE 10. A circle may not be inscribed to the equilateral dodecagon with $k = 4, n = 4$.

Theorem 2.5. *The ratio of surface P_{2n} of the semi-regular polygon $P_{2n}^{a,\delta}$ and the product of multiplication of its side a and the radius of inscribed circle r is equal to the number of sides n of the regular polygon above which it has been constructed, i.e.*

$$\frac{P_{2n}}{ar} = n \tag{16}$$

Proof. From the equation for surface of the semi-regular polygon P_{2n}^{a}

$$P_{2n} = \frac{na^2 \cos\delta cos(\frac{\pi}{n} - \delta)}{\sin\frac{\pi}{n}}$$

and the formula for the radius of the inscribed circle

$$r = r_{2n} = \frac{a\cos\delta\cos(\frac{\pi}{n} - \delta)}{\sin\frac{\pi}{n}}$$

we have the following:

$$P_{2n} = na\left(\frac{a\cos\delta\cos(\frac{\pi}{n} - \delta)}{\sin\frac{\pi}{n}}\right) = anr \Rightarrow \frac{P_{2n}}{ar} = n$$

with n being the number of sides to the regular polygon above which a semi-regular polygon P_{2n}^{a} has been constructed. □

Proposition 2.1. *If the inscribed circle of semi-regular polygon P_{2n}^{a} P is a unit circle, then the ratio of the numerical value of the polygon surface and polygon side is equal to the number of sides n of the corresponding regular polygon above which it has been constructed.*

Proof. It follows from (16), if r=1, i.e. $\frac{P_{2n}}{a} = n$. The following theorem deals with the equilateral semi-regular polygons with the inscribed unit circle. □

Theorem 2.6. *There is no semi-regular equilateral polygon $P_{2n}^{a,\delta}$ with the inscribed unit circle and the side a and the whole number length, i.e. such that $a \in \mathbb{Z}$*

Proof. We have shown (Corollary 2.2) that the radius of the inscribed circle is given through a relation

$$r_{2n} = a\frac{\cos\delta\cos(\frac{\pi}{n} - \delta)}{\sin\frac{\pi}{n}}$$

out of which, for $r = 1$, after the relation processing and shortening, we find that the length of a side of a semi-regular polygon $P_{2n}^{a,\delta}$ is determined with

$$a = \frac{\sin\frac{\pi}{n}}{\cos\delta\cos(\frac{\pi}{n} - \delta)}. \tag{17}$$

If we use that what has been shown for the convex semi-regular equilateral polygons which may be inscribed a circle, that it is applicable that $k = 3, n \geq 3, n \in \mathbb{N}$ and $\delta \in (0, \frac{\pi}{n})$ and that $\sin\alpha \in \mathbb{Q} \Leftrightarrow \alpha = \frac{m\pi}{2}, m \in \mathbb{Z}$ or $\alpha = \frac{\pi}{6}(6l \pm 1), l \in \mathbb{Z}$, as well as that for such values of angle α, $\sin\alpha \in (0, \pm 1, \pm\frac{1}{2})$ we have the following:

$$\sin\pi n \Leftrightarrow \frac{\pi}{n} = \frac{m\pi}{2}, m \in \mathbb{Z} \vee \frac{\pi}{n} = \frac{\pi}{6}(6l \pm 1), l \in \mathbb{Z}$$

From the first equation we get that $n = \frac{2}{m}$, wherefrom $n \in \mathbb{N}$, only for $m = 1$ and $m = 2$. These values do not meet the condition that $n \geq 3$.

From the second equation $\frac{\pi}{n} = \frac{\pi}{6}(6l \pm 1)$ we find that $n = \frac{6}{6l \pm 1}$ and that it represents a natural number $n = 6$ only for $l = 0$. For that value there is a $sin\frac{\pi}{6} = \frac{1}{2}$. Should we replace that value in (17) we obtain the following:

$$a = \frac{1}{2\cos\delta\cos(\frac{\pi}{6} - \delta)}.$$

Since $2\cos\delta\cos(\frac{\pi}{6} - \delta) = \cos(2\delta - \frac{\pi}{6}) + \cos(\frac{\pi}{6})$ the following sequence of inequation is applicable:

$$\begin{aligned} \sqrt{3} &< \cos(2\delta - \frac{\pi}{6}) + \frac{\sqrt{3}}{2} < \frac{2+\sqrt{3}}{2} \Rightarrow \\ \frac{2}{2+\sqrt{3}} &< \frac{1}{\frac{\sqrt{3}}{2} + \cos(2\delta - \frac{\pi}{6})} < \frac{\sqrt{3}}{3} \Leftrightarrow \\ \frac{2}{2+\sqrt{3}} &< a < \frac{\sqrt{3}}{3} \Rightarrow \\ 4 - 2\sqrt{3} &< a < \frac{\sqrt{3}}{3} \end{aligned}$$

the length of side a is not a whole number, what is exactly that what needed to be proven. □

A geometrical construction of the semi-regular polygon which may be inscribed a circle is given within the following theorem.

Theorem 2.7. *For a given radius r of the inscribed circle and angle δ as defined in (5) there is an equilateral semi-regular polygon $P_N^{r,\delta}$, with $N = (k-1)n$ sides for $k = 3, n \geq 3, n \in \mathbb{N}$ which which is defined with those elements and which may be geometrically constructed.*

Proof. Let us presume that the construction of a such semi-regular polygon $P_{2n}^{r,\delta}$ is possible, and that it is presented in Figure 11. Let $\mathcal{C}(O, r)$ be the inscribed circle with its center at point O and with radius $\overline{OK} = r$. We have already shown (Theorem 2.2) that out of all semi-regular polygons $P_{(k-1)n}^{a,\delta}$ a circle may be inscribed only if $k = 3$. Let $A_1B_1A_2B_2\ldots A_nB_n$ be the vertices of a semi-regular polygon constructed above the sides of a regular polygon with vertices $A_1A_2\ldots A_n$, and let neighboring vertices A_i i B_i ,$i = 1, 2, \ldots. n$ have their corresponding interior angles immediately to the vertices in the following sequence: to vertices A_i correspond angles $\alpha = \frac{(n-2)\pi}{n} + 2\delta$, and to vertices B_i correspond angles $\beta = \pi - 2\delta$. Let us take randomly two consecutive vertices of the semi-regular polygon. For the right-angled triangles the following is applicable:

1. For $\triangle OKA_1$ it is: $\angle A_1 = \frac{\alpha}{2} = \frac{(n-2)\pi}{2n} + \delta$, $\angle O = \frac{\pi}{n} - \delta$,$\angle K = \frac{\pi}{2}$,$\overline{OK} = r$

2. For triangle $\triangle OKB_1$ it is: $\overline{OK} = r$,$\angle K = \frac{\pi}{2}$,$\angle B_1 = \pi - 2\delta$ and $\angle O = \delta$ (Figure 11).

Based on the given elements r and δ we can construct the right-angled triangle $\triangle OKB_1$. The intersection of straight line p through points B_1, K with the angle side $\angle O = \frac{\pi}{n} - \delta$ (which may be constructed depending on the number of n sides of the appropriate regular polygon) determines vertex A_1 of right-angled triangle $\triangle OKA_1$. It is with this that we have constructed the side $a = B_1A_1$ of the semi-regular polygon. If we intersect a tangent t_{A_1} from vertex A_1 constructed onto circle $\mathcal{C}(O, r)$ with circle $\mathcal{C}(A_1, a)$ we get vertex B_2. If in that vertex we construct a tangent onto an inscribed circle $\mathcal{C}(O, r)$ the intersection of such tangent and circle $\mathcal{C}(B_2)$ determines vertex $\mathcal{C}(B_2)$. If we proceed further on in the same manner, we may get all the other vertices of the semi-regular equilateral polygon $P_{2n}^{r,\delta}$.

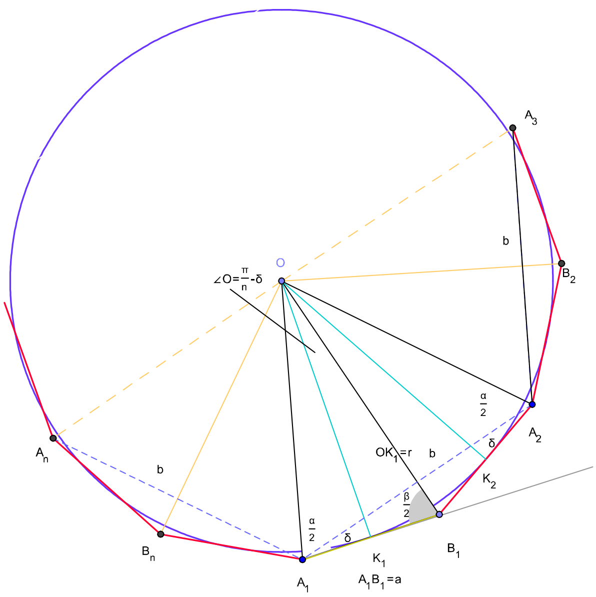

FIGURE 11. Construction of equilateral dodecagon with a given radius of inscribed circle r and angle δ

Construction description: Let there be given radius r and angle δ.

1. We construct a right-angled triangle $\triangle OKB_1$ with the following given elements:

$$\overline{OK} = r, \angle K = \frac{\pi}{2}, \angle O = \delta$$

2. We construct a circle $\mathcal{C}(O, r)$,$\overline{OK} = r$ as an inscribed circle of a semi-regular polygon $P_{2n}^{r,\delta}$.

3. We construct an angle in center O,$\angle O = \frac{\pi}{n} - \delta$, and then construct a right-angled triangle $\triangle OKA_1$, the construction of which determines side $a = B_1A_1$of the semi-regular polygon.

4.We construct a tangent from vertex A_1 to circle $\mathcal{C}(O, r)$ and then we construct vertex B_2, in the following manner $\mathcal{C}(A_1, \overline{B_1A_1} = a) \cap \mathcal{C}(O, r)$.

5. If we repeat the previous procedure this time from vertex B_2 we then get vertex A_2. We then proceed with the same procedure to construct all other vertices. The example above (Figure 12) presents a construction of the semi-regular $P_6^{\delta,r}$ with a given radius $r = 2cm$ of inscribed circle and $\delta = 15°$.

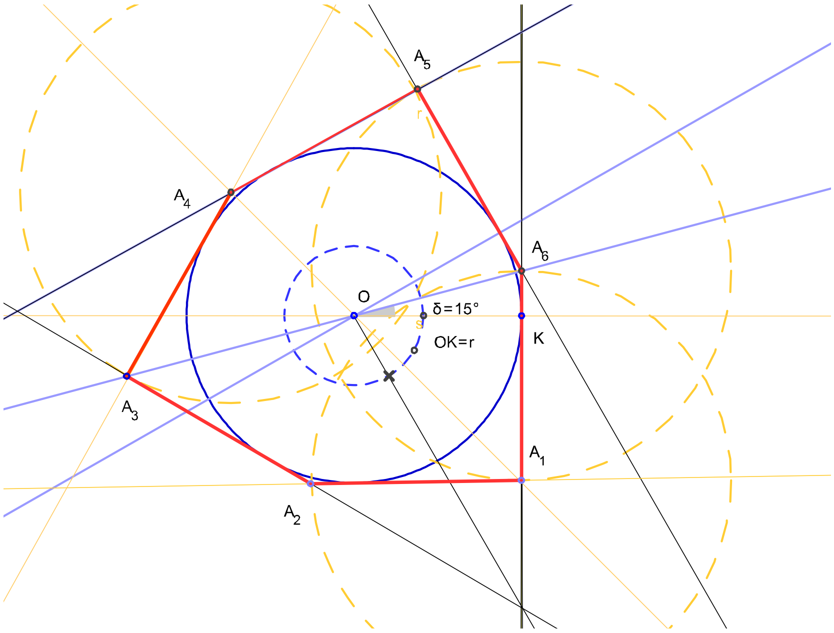

FIGURE 12. Construction $P_6^{\delta,r}, r = 2.5cm, \delta = 15°$

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