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A GEOMETRIC MODEL OF THE FIELD OF COMPLEX NUMBERS

CONSTANTIN LUPU

Abstract. In this paper we will construct a geometric model of the field of complex numbers by using the elementary plane Euclidean geometry notions.

1. Introduction

Considering the set of complex numbers, introduced by the axiomatic method, it can be build a geometric model of this set of numbers by using elementary notions of the plane Euclidean geometry (segment, segment's measure, angle, angle's measure) and also the properties of some elementary geometric transformations (homothety, rotation, symmetry).

Definition 1.1 ([2], [3]). *The triplet* $(K, \oplus, \odot)$, *where* $K \neq \varnothing$, *is called the* ***field of complex numbers*** *if the following conditions (axioms) are true:*

I. $(K, \oplus, \odot)$ *is a commutative field*;
II. The field $(K, \oplus, \odot)$ *is an extension of the field of real numbers* $(\mathbb{R}, +, \cdot)$;
III. There exists an element i *in* K, *with these properties:*
1. $i \odot i = i^2 = -1 \in \mathbb{R}$;
2. *for every element* z *in* K *there exist the real numbers* x *and* y *so that:* $z = x \oplus i \odot y$.

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Remark 1.1. *The field's $(K, \oplus, \odot)$ property of being real numbers field extension has the following meaning: the field $(K, \oplus, \odot)$ contains an subfield $(I, \oplus, \odot)$ which is isomorph with the field $(\mathbb{R}, +, \cdot)$ through a function $\varphi : I \to \mathbb{R}$, so that $\varphi(u) = x$, for every $u \in I$; the function's properties allow the fields $(I, \oplus, \odot)$ and $(\mathbb{R}, +, \cdot)$ to identify: if u, u_1 and u_2 are elements in I so that $\varphi(u) = x$, $\varphi(u_1) = x_2$ and $\varphi(u_2) = x_2$, then $u = x$,*

$$u_1 \oplus u_2 = x_1 + x_2$$

and

$$u_1 \odot u_2 = u_1 \cdot u_2;$$

furthermore, each element $u \in I$ can be replaced by the real number $x = \varphi(u)$ and in the calculations we consider

$$x \oplus y \ (in\ I) = x + y \ (in\ \mathbb{R})$$

and

$$x \odot y \ (in\ I) = x \cdot y \ (in\ \mathbb{R}).$$

Remark 1.2. *In the equalities from the first definition we have*

$$i \odot i = i^2 = -1 \in \mathbb{R}$$

and $z = x \oplus i \odot y$, where the real numbers $-1, x$ and y represent, in fact, the uniquely determined elements $\alpha = i^2$, β and γ from the subfield $(I, \oplus, \odot)$ so that $\varphi(\alpha) = -1$ $(\varphi(\alpha) = \varphi(i^2) = i^2 = -1), \varphi(\beta) = \beta = x$ and $\varphi(\gamma) = \gamma = y$.

Constructing a model of complex numbers set means to specify a construction process of a nonempty K set and to endow it with two composition laws (noted, for example, with the symbols $\oplus$ and $\odot$) so that the triplet $(K, \oplus, \odot)$ must verify the conditions I, II and III from Definition 1.1. In the case of real numbers set, there are known few processes from which we get the models of complex numbers set:

1. *matrix process* (using subsets of the second order quadratic matrix, with real numbers elements; see [1] or [4]);
2. *the quadratic expansion process* or *the process with ordered pairs of real numbers* (based on the quadratic expansion of the field of real numbers $(\mathbb{R}, +, \cdot)$; see [2] or [4]);
3. *the factorisation process* (based on the ring of polynomial factorisation with the $\mathbb{R}[X]$ with the real coefficients through its prime ideal; see [5]);

We will build, in the following lines, a geometric model of complex numbers set by using the Euclidean plan's elementary geometry notions.

In the Euclidean plane ε_2, which is endowed with a Cartesian system of coordinates (XOY), we consider the set $K = \{z \mid z = \overrightarrow{OM},\ M \in \varepsilon_2\}$. In this set we are consider:

- the element (the null vector) $\overrightarrow{OO}$ from K is noted with 0_K;

- for each element $z = \overrightarrow{OM}$ in $K \setminus \{0_K\}$ is associated with the unique real number $t \in 0; 2\pi)$, where $t = \arg z = \mu\left(\widehat{OX; OM}\right)$, where the angle $\left(\widehat{OX; OM}\right)$ is positively oriented (see Figure 1);

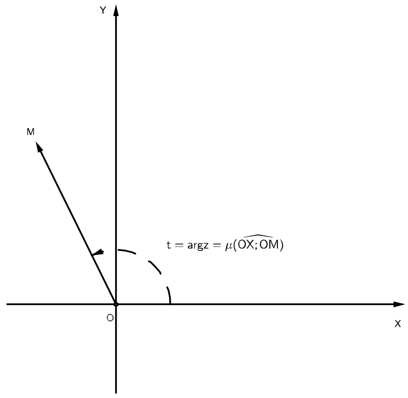

Figure 1

- the number $t = \arg z$ is called *the reduced argument* and the set $Argz = \{\arg z + 2k\pi, k \in \mathbb{Z}\}$ is called *the extended argument* for the element z;

- the element $0_K = \overrightarrow{OO}$ has the reduced argument undetermined, which means $\arg 0_K = t$, for all $t \in [0; 2\pi)$;

The notion of argument allows the next simple remarks for the elements $z_1 = \overrightarrow{OM}$ and $z_2 = \overrightarrow{ON}$ in $K \setminus \{0_K\}$:

a) z_1 and z_2 have *the same direction and the same way* if and only if $\arg z_1 = \arg z_2$ (see Figure 2);

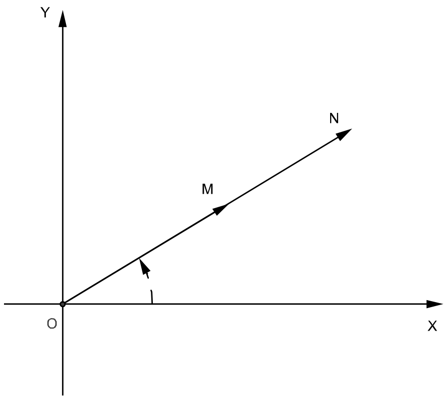

Figure 2

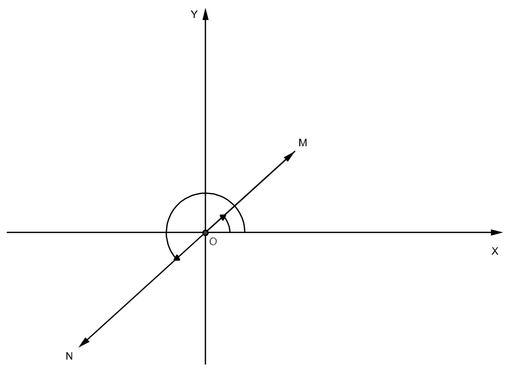

Figure 3

b) z_1 and z_2 have *opposite directions* if and only if $\arg z_2 - \arg z_1 = \pi \pmod{2\pi}$ (see Figure 3);

c) $z_1 = z_2$ if and only if $|z_1| = |z_2|$ and $\arg z_1 = \arg z_2$;

Also, if $z = \overrightarrow{OM}$ is an element from $K \setminus \{0_K\}$, then:

d) $M \in OX_+ \Leftrightarrow \arg z = 0$ and $M \in OX_- \Leftrightarrow \arg z = \pi$;

e) $M \in OY_+ \Leftrightarrow \arg z = \frac{\pi}{2}$ and $M \in OY_- \Leftrightarrow \arg z = \frac{3\pi}{2}$.

We will remind the definitions of some elementary geometric transformations that will be used in the construction of the proposed model:

Let be the Euclidean plane ε_2, which is endowed with a Cartesian system of coordinates (XOY).

Definition 1.2. *A homothety of center O and ratio (power) $k \in \mathbb{R}_+^*$ is the geometric transformation $\phi : \varepsilon_2 \to \varepsilon_2$ which satisfies the following conditions:*

1) $\phi(O) = O$, *were O is the origin of the Cartesian system of coordinates* (XOY);
2) *if $M \neq O$ is a point from ε_2 and $M' = \phi(M)$, then:*
a) the point M' is situated on the ray $(OM$;
b) $|OM'| = k \cdot |OM|$. (see Figure 4).

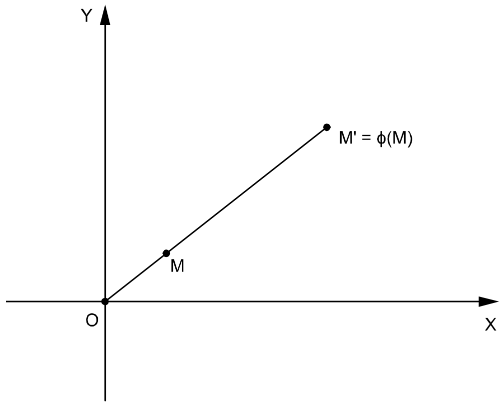

Figure 4

Definition 1.3. *The rotation of center O and angle positively oriented θ is the geometric transformation $\Re : \varepsilon_2 \to \varepsilon_2$ which satisfies the following conditions:*

1) $\Re(O) = O$, *where O is the origin of the cartesian system of coordinates* (XOY);
2) *if $M \neq O$ is a point from ε_2 and $M' = \Re(M)$, then:*
a) $\mu\left(\widehat{MOM'}\right) = \theta$;
b) $|OM| = |OM'|$. (see Figure 5).

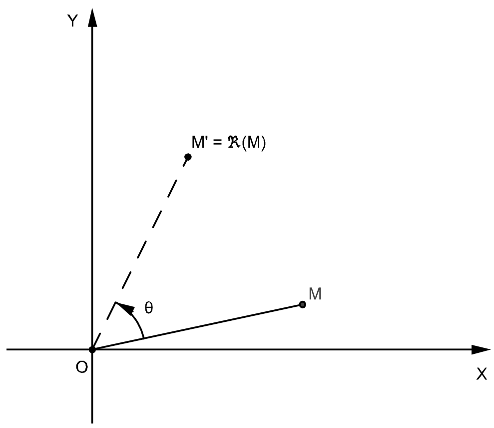

Figure 5

Definition 1.4. *The symmetry with respect to a line d is the geometric transformation $S_d : \varepsilon_2 \to \varepsilon_2$ which satisfies the following conditions:*

1) $S_d(A) = A, \forall A \in d$;
2) *if* $M \notin d$ *and* $M' = S_d(M)$, *then:*
a) $MM' \perp d$
b) $|MP| = |PM'|$, where $MM' \cap d = \{P\}$. (see Figure 6).

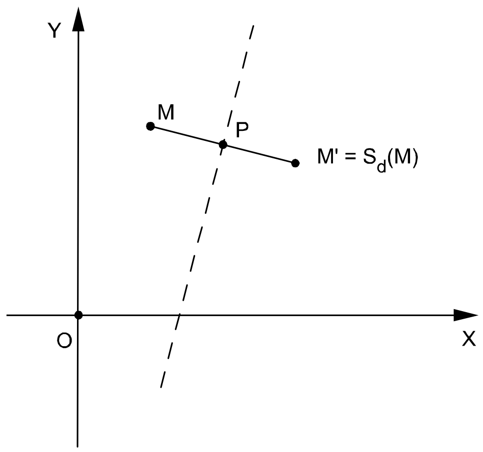

Figure 6

2. Main Result

Theorem 2.1. *In the Euclidean plane ε_2 endowed with a Cartesian system of coordinates (XOY), we consider the set of vectors $K = \left\{ z | z = \overrightarrow{OM},\ M \in \varepsilon_2 \right\}$ and the operations $\oplus : K \times K \to K$, $\odot : K \times K \to K$ where its are defined as follows:*

1. the operation $\oplus$ represents the usual addition of fixed vectors

(1) $$from\ plane\ (XOY),\ with\ the\ application\ point\ at\ the\ origin\ O;$$

2. a) if $z_1 = \overrightarrow{OM}$ and $z_2 = \overrightarrow{ON}$ are elements from $K \setminus \{0_K\}$, then $z_1 \odot z_2 = z = \overrightarrow{OP}$, where

$$|z| = \left| \overrightarrow{OP} \right| = |z_1| \cdot |z_2|$$

and

(2) $$\arg z = (\arg z_1 + \arg z_2) \pmod{2\pi}$$

b) if $z_1 = 0_K$ and $z_2 \neq 0_K$, or $z_1 \neq 0_K$ and $z_2 = 0_K$, or $z_1 = z_2 = 0_K$, then

(3) $$z_1 \odot z_2 = 0_K$$

Theorem 2.2. *The triplet* $(K, \oplus, \odot)$ *represents a model of the set of the complex numbers.*

Proof. We will show that the triplet $(K, \oplus, \odot)$ verify the conditions I,II and III from Definition 1.1. First ,we indicate the geometrical procedures for the obtaining of the sum and the product of two elements from $K\backslash\{0_K\}$. Let $z_1 = \overrightarrow{OM}$ and $z_2 = \overrightarrow{ON}$ are elements of the set $K\backslash\{0_K\}$. Then the element $z = z_1 \oplus z_2$ is obtained by parallelogram rule (see Figure 7),

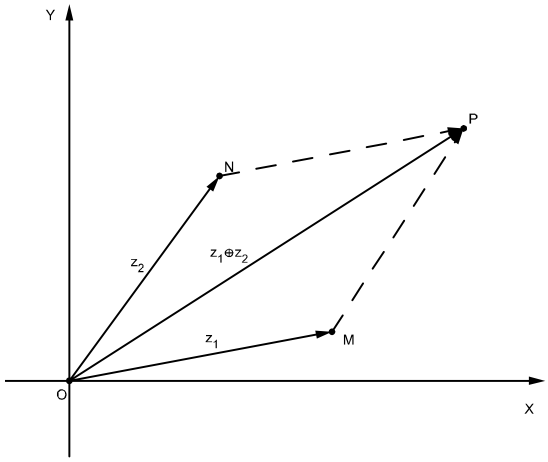

Figure 7

and the element $w = z_1 \odot z_2$ it is obtained in two steps, as follows:

- we consider the homothety ϕ of center O and ratio $k = |z_1|$, and we obtain the point $N' = \phi(N)$ so that $N' \in (ON$ and $|ON'| = |z_1| \cdot |ON| = |z_1| \cdot |z_2|$;

- we consider the rotation $\Re$ of center O and angle of measure $\arg z_1$, and we obtain the point $Q = \Re(N')$ so that $|OQ| = |ON'| = |z_1| \cdot |z_2|$ and $\mu\left(\widehat{N'OQ}\right) = \arg z = \theta$. The vector $w = \overrightarrow{OQ}$ represents the element $z_1 \odot z_2$ because $|w| = |z_1| \cdot |z_2|$ and $\arg w = (\arg z_1 + \arg z_2)(\bmod 2\pi)$ (see Figure 8).

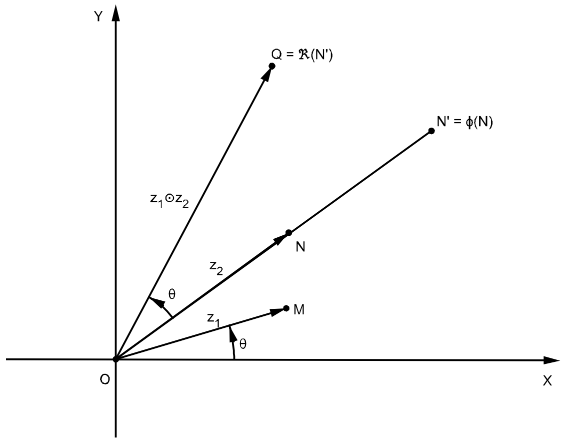

Figure 8

So, if we represent in the plane ε_2 the elements z_1 and z_2 from $K\backslash\{0_K\}$, then the element $z_1 \oplus z_2$ is obtained by the rule of the parallelogram and the lement $z_1 \odot z_2$ is obtained by composing the rotation $\Re$ by the center O and angle $\theta = \arg z_1$ with the homotety by center O and ratio $k = |z_1|$.

I. The triplet $(K, \oplus, \odot)$ is a commutative field.

(K_1). The composition laws $\oplus$ and $\odot$ are always defined on the set K: if z_1 and z_2 are elements from K, then, according to rules (1), (2) and (3), we deduce that $z_1 \oplus z_2$ and $z_1 \odot z_2$ are elements from K;

(K_2). The composition laws $\oplus$ and $\odot$ are commutative: if $z_1, z_2 \in K$, then

$$z_1 \oplus z_2 = z_2 \oplus z_1$$

(property of usual vector addition); considering $u = z_1 \odot z_2$ and $v = z_2 \odot z_1$, we deduce that

$$|u| = |z_1| \cdot |z_2| = |z_2| \cdot |z_1| = |v|$$

and

$$\arg u = (\arg z_1 + \arg z_2) \bmod 2\pi = (\arg z_2 + \arg z_1)\,(\bmod 2\pi = \arg v)\,,$$

so $u = v$.

(K_3). The laws $\oplus$ and $\odot$ are associative: if z_1, z_2, $z_3 \in K$, then

$$(z_1 \oplus z_2) \oplus z_3 = z_1 \oplus (z_2 \oplus z_3)$$

(the associativity of usual vector addition); considering $u = (z_1 \odot z_2) \odot z_3$ and $v = z_1 \odot (z_2 \odot z_3)$ and applying the laws (1), (2) and (3) we easily conclude that ų$= v$;

(K_4). 1) The element $0_K = \overrightarrow{OO}$ from K is neutral element with respect to the law $\oplus$. If $z = \overrightarrow{OM} \in K$, then $z \oplus 0_K = 0_K \oplus z = z$;

2) The element $1_K = \overrightarrow{OU}$ from K, with $U \in OX_+$ and $\left|\overrightarrow{OU}\right| = 1$ is neutral element with respect to the law $\odot$. If $z \in K$ and $z' = z \odot 1_K$, $z'' = 1_K \odot z$, then

$$\left|z'\right| = |z| \cdot |1_K| = |z|\,, \left|z''\right| = |1_K| \cdot |z| = |z|\,,$$

$$\arg z' = (\arg z + \arg 1_K) \bmod 2\pi = (\arg z + 0) \bmod 2\pi = \arg z$$

and

$$\arg z'' = (\arg 1_K + \arg z) \bmod 2\pi = (0 + \arg z) \bmod 2\pi = \arg z.$$

It follows that $z' = z'' = z$, hence $z \odot 1_K = 1_K \odot z = z$.

(K_5). 1) For every $z = \overrightarrow{OM} \in K$ there exists an opposite element $\ominus z = \overrightarrow{OM'} \in K$, with $\left|\overrightarrow{OM'}\right| = \left|\overrightarrow{OM}\right|$ and $\arg \overrightarrow{OM'} = (\arg z + \pi) \bmod 2\pi$ (see Figure 9).

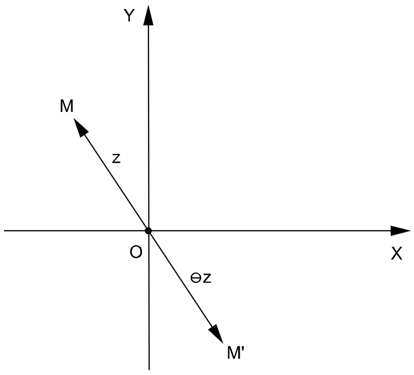

Figure 9

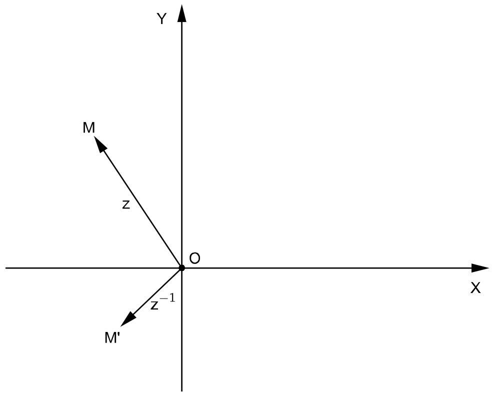

Figure 10

If $z \in K$ and $z = \overrightarrow{OM}$, $\ominus z = \overrightarrow{OM'}$ with $|z| = |\ominus z|$ and

$$\arg \ominus z = (\arg z + \pi) \bmod 2\pi,$$

then the vectors $\overrightarrow{OM}$ and $\overrightarrow{OM'}$ are collinear, they have tha same length but opposite directions, so

$$z \oplus (\ominus z) = (\ominus z) \oplus z = 0_K.$$

2) For every $z = \overrightarrow{OM} \in K \backslash \{0_K\}$ there exists an inverse element $z^{-1} = \overrightarrow{OM'} \in K$, with

$$\left|z^{-1}\right| = \frac{1}{|z|}$$

and

$$\arg z^{-1} = (2\pi - \arg z) \bmod 2\pi$$

(see Figure 10).

If $w = z \odot z^{-1}$, then

$$|w| = |z| \cdot \left|z^{-1}\right| = |z| \cdot \frac{1}{|z|} = 1$$

and

$$\arg w = \left(\arg z + \arg z^{-1}\right) \bmod 2\pi = (2\pi) \bmod 2\pi = 0,$$

we get that $w = 1_K$. Similarly we find that $z^{-1} \odot z = 1_K$.

(K_6) The law $\odot$ is distributive over the law $\oplus$. Let the vectors $z_1 = \overrightarrow{OM_1}$, $z_2 = \overrightarrow{OM_2}$ and $z = \overrightarrow{OM}$ be from $K \backslash \{0_K\}$. We show that

$$z \odot (z_1 \oplus z_2) = (z \odot z_1) \oplus (z \odot z_2).$$

Representing in the Euclidean plane ε_2 the vectors $z = \overrightarrow{OM}$, $z_1 = \overrightarrow{OM_1}$, $z_2 = \overrightarrow{OM_2}$ and $z_1 \oplus z_2 = \overrightarrow{OP}$, it is obvious that the quadrilateral OM_1PM_2 is a parallelogram (see Figure 11).

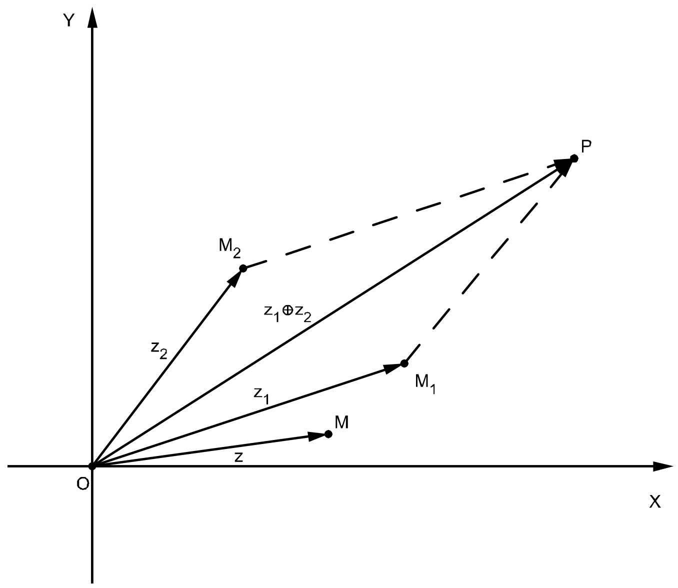

Figure 11

a) Considering the homothety ϕ of centre O and power $|z|$, if $M_1^{'} = \phi(M_1)$, $M_2^{'} = \phi(M_2)$ and we take the parallelogram $OM_1^{'}QM_2^{'}$ (see Figure12) with $\left|OM_1^{'}\right| = |z| \cdot |z_1|$, $\left|M_1^{'}Q\right| = |z| \cdot |z_2|$ we deduce that the triangles OM_1P (Figure 11) and $OM_1^{'}Q$ (Figure 12) are similar.

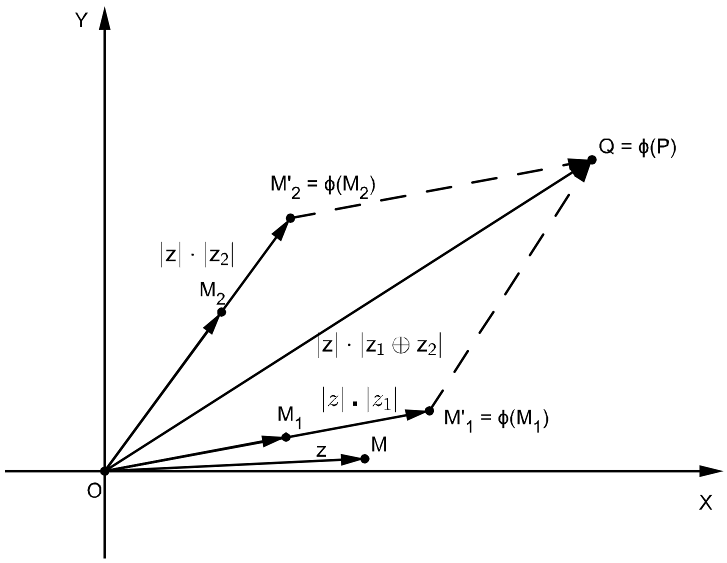

Figure 12

We find that

$$\frac{|OP|}{|OQ|} = \frac{|z_1|}{|z| \cdot |z_1|}$$

i.e.

$$\frac{|z_1 \oplus z_2|}{|OQ|} = \frac{1}{|z|},$$

so $|OQ| = |z| \cdot |z_1 \oplus z_2|$; in these conditions we obtain that $Q = \phi(P)$;

b) We consider the rotation $\Re$ with the centre O and angle $\theta = \arg z$. If $M_1^{"} = \Re(M_1^{'})$, $Q^{'} = \Re(Q)$ and $M_2^{"} = \Re(M_2^{'})$ it results that the quadrilateral $OM_1^{"}Q^{'}M_2^{"}$ is a parallelogram (see Figure 13), with $\overrightarrow{OM_1^{"}} = z \odot z_1$, $\overrightarrow{OQ^{'}} = z \odot (z_1 \oplus z_2)$ and $\overrightarrow{OM_2^{"}} = z \odot z_2$.

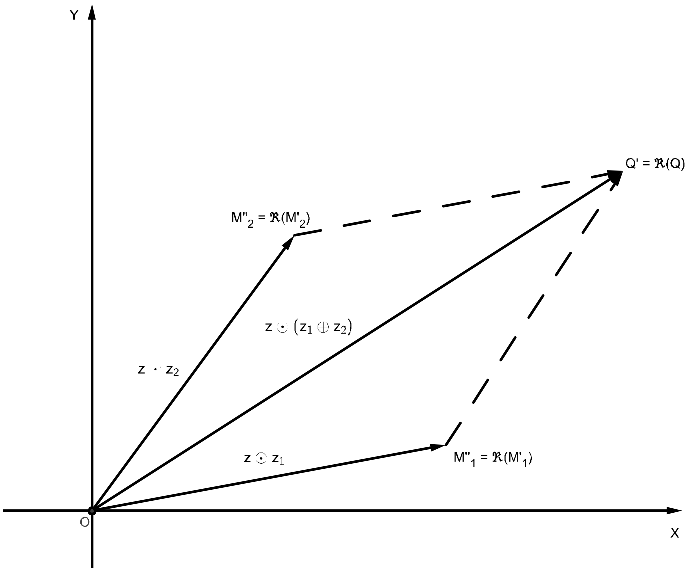

Figure 13

Applying the addition rule for the vectors $\overrightarrow{OM_1^{"}}$ and $\overrightarrow{OM_2^{"}}$, it follows that $\overrightarrow{OQ'} = (z \odot z_1) \oplus (z \odot z_2)$, so

$$\overrightarrow{OQ'} = z \odot (z_1 \oplus z_2) = (z \odot z_1) \oplus (z \odot z_2).$$

II. *The field* $(K, \oplus, \odot)$ *is an extension of the field of real numbers* $(\mathbb{R}, +, \cdot)$.

a) We consider the set $I \subset K$, with $I = \{z \in K \mid z = \overrightarrow{OM},\ M \in OX\}$. We see that z is an element from I if and only if $\arg z = 0$ or $\arg z = \pi$ (excepting the element 0_K which has undertermined argument) We simply show that the triplet $(I, \oplus, \odot)$ is an subfield of the field K because the next conditions are satisfied:

1) If z and v are elements from I, then $z \ominus v$ is an element from I (the property results by applying the usual rule of vector addition);

2) If $z = \overrightarrow{OM}$ and $v = \overrightarrow{ON} \neq 0_K$ are elements from I, then the element $z \odot v^{-1}$ is from I; indeed, since $v \in I$, v^{-1} is an element from I because

$$\arg v^{-1} = (2\pi - \arg v) \bmod 2\pi = \begin{cases} (2\pi - 0) \bmod 2\pi = 0, & \text{if } N \in OX_+ \\ (2\pi - \pi) \bmod 2\pi = \pi & \text{if } N \in OX_- \end{cases}$$

Then we have

$$\arg z \odot v^{-1} = (\arg z + \arg v^{-1}) \bmod 2\pi = 0 \text{ or } \pi,$$

so $z \odot v^{-1} \in I$.

b) Let $\varphi : (I, \oplus, \odot) \longrightarrow (\mathbb{R}, +, \cdot)$ be a function defined as follows: if $z = \overrightarrow{OM}$ is an element from I, then

$$\varphi(z) = x = \begin{cases} |z| = \left|\overrightarrow{OM}\right|, & \text{if } M \in OX_+ \\ 0, & \text{if } M = O \\ -|z| = -\left|\overrightarrow{OM}\right|, & \text{if } M \in OX_- \end{cases}$$

The function φ is an isomorphism between the fields $(I, \oplus, \odot)$ and $(\mathbb{R}, +, \cdot)$ because the next conditions are fulfilled:

1) φ is a bijective function (acount of the axiom of construction of a segment with a precised length and of φ);

2) $\varphi(z_1 \oplus z_2) = \varphi(z_1) + \varphi(z_2)$,

3) $\varphi(z_1 \odot z_2) = \varphi(z_1) \cdot \varphi(z_2)$ for any z_1 and z_2 from I.

Let's assume, without restrict the generality, that $z_1 = \overrightarrow{OM}$ and $z_2 = \overrightarrow{ON}$, are elements from I with the property that $M, N \in OX_+$. If $|z_1| = x_1 > 0$ and $|z_2| = x_2 > 0$, then

$$\varphi(z_1 \oplus z_2) = \varphi(\overrightarrow{OM} \oplus \overrightarrow{ON}) = \varphi(\overrightarrow{OP}),$$

with $P \in OX_+$ and $\left|\overrightarrow{OP}\right| = \left|\overrightarrow{OM}\right| + \left|\overrightarrow{ON}\right| = x_1 + x_2$ (the rule of addition of the vectors with the extremities on the axis OX). Therefore,

$$\varphi(z_1 \oplus z_2) = \varphi(z_1) + \varphi(z_2).$$

The property 2) can be proved in the same manner and in the cases $M \in Ox_+$, $N \in Ox_-$, or $M \in Ox_-$ and $N \in Ox_-$; also, in these conditions, if

$$z_1 \odot z_2 = \overrightarrow{OM} \odot \overrightarrow{ON} = \overrightarrow{OQ},$$

with $\left|\overrightarrow{OQ}\right| = \left|\overrightarrow{OM}\right| \cdot \left|\overrightarrow{ON}\right| = x_1 \cdot x_2$ and

$$\arg \overrightarrow{OQ} = (\arg \overrightarrow{OM} + \arg \overrightarrow{ON}) \bmod 2\pi = 0,$$

it follows that:

$$\varphi(z_1 \odot z_2) = \varphi(\overrightarrow{OQ}) = x_1 \cdot x_2 = \varphi(z_1) \cdot \varphi(z_2).$$

Therefore that any element $z = \overrightarrow{OM}$ from I can be identified through the real number $x = \varphi(z)$ (it's written simplified $z = x$) and sums, respective products of the types $z_1 \oplus z_2$ and $z_1 \odot z_2$ from I its can identify with sums, respective products of the types $x_1 + x_2$, respective $x_1 \cdot x_2$ from $\mathbb{R}$ (we written $z_1 \oplus z_2 = x_1 + x_2$ and $z_1 \odot z_2 = x_1 \cdot x_2$). In conclusion , the field $(K, \oplus, \odot)$ it is an extesion of the field of the real numbers $(\mathbb{R}, +, \cdot)$, the set $\mathbb{R}$ thus considered that a subset of K.

III. There is an element i in K with the properties:

1. $i \odot i = i^2 = -1 \in \mathbb{R}$;

2. For every element z from K, it can be indicated the real numbers x and y, so that $z = x \oplus i \odot y$.

We consider the element i from K with $i = \overrightarrow{OV}$, $V \in OY_+$ so that $|i| = \left|\overrightarrow{OV}\right| = 1$ and, obviously, $\arg i = \frac{\pi}{2}$.

1. Let be $\overrightarrow{OP} = i \odot i$; then $\left|\overrightarrow{OP}\right| = |i \odot i| = |i| \cdot |i| = 1 \cdot 1 = 1$, and

$$\arg \overrightarrow{OP} = (\arg i + \arg i) \bmod 2\pi = \frac{\pi}{2} + \frac{\pi}{2} = \pi,$$

the point P is on the axis OX_-. We have

$$\varphi(i \odot i) = \varphi(i^2) = \varphi(\overrightarrow{OP}) = -\left|\overrightarrow{OP}\right| = -1,$$

therefore $i^2 = -1$;

2. Let be $z = \overrightarrow{OM} \in K\backslash\{0_K\}$. We will demostrate that the real numbers x and y are existing, so that $z = x \oplus i \odot y$;

a) We will assume that the extremity M of the vector $z = \overrightarrow{OM}$ is positioned in the first or the second quadrant (see Figure 14) and let M_1 and M_2 are the projections of M on the axis OX, respectively OY.

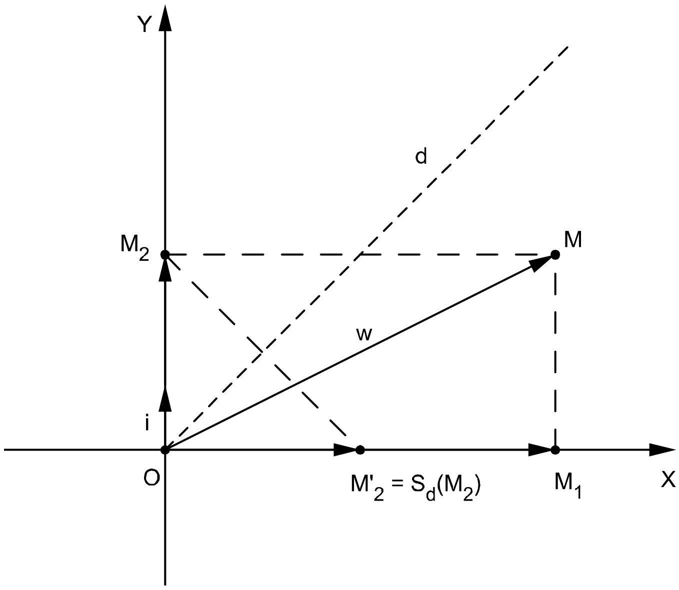

Figure 14

According to the rule of addition of the vectors, it results that:

$$\overrightarrow{OM} = \overrightarrow{OM_1} \oplus \overrightarrow{OM_2} \tag{4}$$

Since, the point M_1 is situated on the axis OX, the vector $\overrightarrow{OM_1}$ it is identified through the real number $x = \pm\left|\overrightarrow{OM_1}\right|$, as $M_1 \in OX_+$ or $M_1 \in OX_-$, so

$$\overrightarrow{OM_1} = x. \tag{5}$$

Now we consider the simmetry S_d, where d represents the first bisetrix and let be $M_2' = S_d(M_2) \in OX$ (see Figure14), so $\left|\overrightarrow{OM_2'}\right| = \left|\overrightarrow{OM_2}\right|$. In theese conditions, $\overrightarrow{OM_2} = i \odot \overrightarrow{OM_2'}$ because $\left|\overrightarrow{OM_2}\right| = |i| \cdot \left|\overrightarrow{OM_2'}\right| = \left|\overrightarrow{OM_2'}\right|$, and

$$(\arg i + \arg \overrightarrow{OM_2'}) \bmod 2\pi = \frac{\pi}{2} + 0 = \frac{\pi}{2} = \arg \overrightarrow{OM_2}.$$

Since, the vector $\overrightarrow{OM_2'}$, is indetified through the real number $y = \left|\overrightarrow{OM_2'}\right| = \left|\overrightarrow{OM_2}\right|$, it results that

(6) $$\overrightarrow{OM_2} = i \odot y.$$

Replacing (5) and (6) in (4), we deduce that:

$$z = \overrightarrow{OM} = \overrightarrow{OM_1} \oplus \overrightarrow{OM_2} = x \oplus i \odot y.$$

b) We assume now that the extremity M of $z = \overrightarrow{OM}$ is positioned in the third or the fourth quadrant (see Figure15). Let $M_1$and M_2 be the projections of the point M on the axis OX and OY respectively**.**

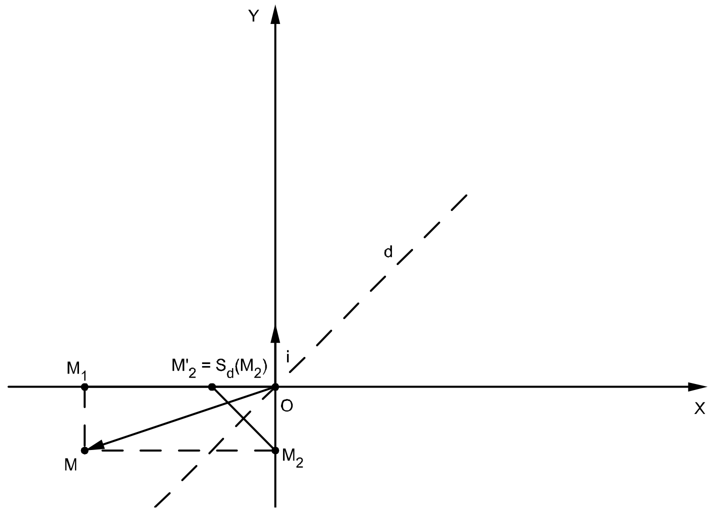

Figure 15

Considering $M_2' = S_d(M_2)$, we deduce that $M_2' \in OX_-$ and $\overrightarrow{OM_2'}$ vector is identified through the real number $y < 0$, with $y = -\left|\overrightarrow{OM_2'}\right| = -\left|\overrightarrow{OM_2}\right|$.
But $\overrightarrow{OM_2} = i \odot \overrightarrow{OM_2'}$ because $\left|\overrightarrow{OM_2}\right| = |i| \cdot \left|OM_2'\right|$ and

$$\arg(i \odot \overrightarrow{OM_2'}) = (\arg i + \arg \overrightarrow{OM_2'}) \bmod 2\pi = (\frac{\pi}{2} + \pi) \bmod 2\pi = \frac{3\pi}{2}.$$

Since $\overrightarrow{OM_2'} = y$ (through φ isomorphism), it results that $\overrightarrow{OM_2} = i \odot y$. Therefore, replacing in the equality $\overrightarrow{OM} = \overrightarrow{OM_1} + \overrightarrow{OM_2}$, $\overrightarrow{OM_1}$ with x and $\overrightarrow{OM_2}$ with $i \odot y$, we obtain $z = \overrightarrow{OM} = \overrightarrow{OM_1} \oplus \overrightarrow{OM_2} = x \oplus i \odot y$. □

Being met the first, second, and third conditions from the definition of the complex numbers set, we deduce that the ensemble $(K, \oplus, \odot)$ is a model of the complex numbers set and they contain the follow conditions:

- K set is noted with the symbol $\mathbb{C}$;
- the operation $\oplus$ and $\odot$ are noted with the classic symbols "+" and "·";
- the triplet ($\mathbb{C}$,+,·) is named the **complex numbers set;**
- the elements of $\mathbb{C}$ set are named **complex numbers;**
- the field ($\mathbb{C}$, +, ·) is named the **complex numbers field**

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"GRIGORE ANTIPA" NATIONAL COLLEGE
HENRI COANDĂ 7, BACĂU, ROMANIA
E-mail address: lavinialupu26@yahoo.com