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BERTRAND AND MANNHEIM PARTNER D-CURVES ON PARALLEL SURFACES IN MINKOWSKI 3-SPACE

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Abstract. In this paper, by considering parallel surfaces we study Bertrand and Mannheim partner D-curves in Minkowski 3-space E_1^3. We find the images of two curves which lie on two different surfaces and satisfy the conditions to be Bertrand D-curve or Mannheim D-curve in E_1^3. Then we obtain relationships between Bertrand and Mannheim partner D-curves and their image curves.

1. Introduction

In local differential geometry, the properties of a curve which is related to another curve are interesting and fascinating subject of the curve theory. The well-known examples of these curve pairs are Bertrand curves, involute-evolute curves and Mannheim partner curves. These special curves have been studied by many mathematicians in different spaces [3,4,5,6,11,13,17]. Moreover, some new definitions of special curve pairs have been given by Kazaz and et all. They have given two new types of these curves which lie on regular surfaces fully and called Bertrand partner D-curves and Mannheim partner D-curves [7,8]. Using the Darboux frames of these curves they have obtained characterizations of these new curve pairs. They have also considered the same special curve pairs in Minkowski 3-space and investigated the different conditions according to the Lorentzian casual characters of the curves and surfaces [9,10].

Moreover, analogue to the associated curves, similar relationships can be constructed between regular surfaces. For example, a surface and another surface which have constant distance with the reference surface along its surface normal have a relationship between their parametric representations. Such surfaces are called parallel surface [2]. By this definition, it is convenient to carry the points of a surface to the points of another surface. Since the curves are the set of points, then the curves lying fully on a reference surface can be carry to another surface.

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In this study, we consider the images of Bertrand and Mannheim partner D-curves on parallel surfaces in Minkowski 3-space E_1^3. First, we obtain the frames of image curves of these curve pairs on parallel surfaces. Then, we investigate the relationships between reference curves and their images.

2. Preliminaries

Let E_1^3 be a Minkowski 3-space with natural Lorentz Metric

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. According to this metric, in E_1^3 an arbitrary vector $\vec{v} = (v_1, v_2, v_3)$ can have one of three Lorentzian causal characters; it can be spacelike if $\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$ and null (lightlike) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ is spacelike, timelike or null (lightlike), if all of its velocity vectors $\vec{\alpha'}(s)$ are spacelike, timelike or null (lightlike), respectively [12]. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. For any vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in E_1^3, the vector product of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \times \vec{y} = (x_2 y_3 - x_3 y_2,\; x_1 y_3 - x_3 y_1,\; x_2 y_1 - x_1 y_2).$$

A surface in the Minkowski 3-space E_1^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [1].

Let S be an oriented surface in E_1^3 and let consider a non-null curve $\alpha(s)$ lying fully on S. Since the curve $\alpha(s)$ is also in the space, there exists a Frenet frame $\{T, N, B\}$ along the curve where T is unit tangent vector, N is principal normal vector and B is binormal vector, respectively. Moreover, since the curve $\alpha(s)$ lies on the surface S there exists another frame along the curve $\alpha(s)$. This frame is called Darboux frame and denoted by $\{T, Y, Z\}$ which gives us an opportunity to investigate the properties of the curve according to the surface. In this frame T is the unit tangent of the curve, Z is the unit normal of the surface S along $\alpha(s)$ and Y is a unit vector given by $Y = \pm Z \times T$. Since the unit tangent T is common in both Frenet frame and Darboux frame, the vectors N, B, Y and Z lie on the same plane. So that the relations between these frames can be given as follows:

If the surface S is an oriented timelike surface, the relations between the frames can be given as follows

If the curve $\alpha(s)$ is timelike

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

If the curve $\alpha(s)$ is spacelike

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

If the surface S is an oriented spacelike surface, then the curve $\alpha(s)$ lying on S is a spacelike curve. So, the relations between the frames can be given as follows

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

In all cases, φ is the angle between the vectors Y and N [15,16].

According to the Lorentzian causal characters of the surface S and the curve $\alpha(s)$ lying on S, the derivative formulae of the Darboux frame can be changed as follows:

i) If the surface S is a timelike surface, then the curve $\alpha(s)$ lying on S can be a spacelike or a timelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by

(1)
$$\begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & k_g & -\varepsilon k_n \\ k_g & 0 & \varepsilon t_r \\ k_n & t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix}, \quad \langle T,T\rangle = \varepsilon = \pm 1, \quad \langle Y,Y\rangle = -\varepsilon, \quad \langle Z,Z\rangle = 1.$$

ii) If the surface S is a spacelike surface, then the curve $\alpha(s)$ lying on S is a spacelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by

(2)
$$\begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_r \\ k_n & t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix}, \quad \langle T,T\rangle = 1, \quad \langle Y,Y\rangle = 1, \quad \langle Z,Z\rangle = -1.$$

In these formulae k_g, k_n and t_r are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively and the prime shows the derivatives with respect to arc length s [15,16].

In the differential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface S the followings are well-known, (See [14]),

i) $\alpha(s)$ is a geodesic curve $\Leftrightarrow k_g = 0$,
ii) $\alpha(s)$ is an asymptotic line $\Leftrightarrow k_n = 0$,
iii) $\alpha(s)$ is a principal line $\Leftrightarrow t_r = 0$.

Definition 2.1 ([10]). *Let S and S_1 be oriented surfaces in Minkowski 3-space E_1^3 and let consider the arc-length parameter curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1, respectively. Denote the Darboux frames of $\alpha(s)$ and $\alpha_1(s_1)$ by $\{T, Y, Z\}$ and $\{T_1, Y_1, Z_1\}$, respectively. If there exists a corresponding relationship between the curves $\alpha(s)$ and $\alpha_1(s_1)$ such that, at the corresponding points of the curves, direction of the Darboux frame element Y of $\alpha(s)$ coincides with direction of the Darboux frame element Y_1 of $\alpha_1(s_1)$*

then $\alpha(s)$ is called a Bertrand D-curve, and $\alpha_1(s_1)$ is a Bertrand partner D-curve of $\alpha(s)$. Then, the pair $\{\alpha, \alpha_1\}$ is said to be a Bertrand D-pair.

If both the surface S and the curve $\alpha(s)$ lying on S are spacelike then, there are two cases; first one is that both the surface S_1 and the curve $\alpha_1(s_1)$ fully lying on S_1 are spacelike. In this case we say that the pair $\{\alpha, \alpha_1\}$ is a Bertrand $D-$pair of the type 1. The second case is that both the surface S_1 and the curve $\alpha_1(s_1)$ fully lying on S_1 are timelike. Then the pair $\{\alpha, \alpha_1\}$ is called a Bertrand $D-$pair of the type 2. If both the surface S and the curve $\alpha(s)$ lying on S are timelike then, there are two cases; one is that both the surface S_1 and the curve $\alpha_1(s_1)$ fully lying on S_1 are timelike. In this case we say that the pair $\{\alpha, \alpha_1\}$ is a Bertrand D-pair of the type 3. The other case is that both the surface S_1 and the curve $\alpha_1(s_1)$ fully lying on S_1 are spacelike then the pair $\{\alpha, \alpha_1\}$ is a Bertrand D-pair of the type 4. If the surface S is timelike and the curve $\alpha(s)$ lying on S is spacelike then the surface S_1 is timelike and the curve $\alpha_1(s_1)$ fully lying on S_1 is spacelike. In this case we say that the pair $\{\alpha, \alpha_1\}$ is a Bertrand D-pair of the type 5 [10].

Definition 2.2 ([9]). *Let S and S_1 be oriented surfaces in Minkowski 3-space E_1^3 and let consider the arc-length parameter curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1, respectively. Denote the Darboux frames of $\alpha(s)$ and $\alpha_1(s_1)$ by $\{T, Y, Z\}$ and $\{T_1, Y_1, Z_1\}$, respectively. If there exists a corresponding relationship between the curves $\alpha(s)$ and $\alpha_1(s_1)$ such that, at the corresponding points of the curves, direction of the Darboux frame element Y of $\alpha(s)$ coincides with direction of the Darboux frame element Z_1 of $\alpha_1(s_1)$ then $\alpha(s)$ is called a Mannheim D-curve, and $\alpha_1(s_1)$ is a Mannheim partner D-curve of $\alpha(s)$. Then, the pair $\{\alpha, \alpha_1\}$ is said to be a Mannheim D-pair.*

If both the surface S and the curve $\alpha(s)$ lying on S are spacelike then, there are two cases; first one is that the surface S_1 is a timelike surface and the curve $\alpha_1(s_1)$ fully lying on S_1 is spacelike. In this case we say that the pair $\{\alpha, \alpha_1\}$ is a Mannheim D-pair of the type 1. The second one is that both the surface S_1 and the curve $\alpha_1(s_1)$ fully lying on S_1 are timelike. In this case we say that the pair $\{\alpha, \alpha_1\}$ is a Mannheim $D-$pair of the type 2. If both the oriented surface S and $\alpha(s)$ lying on S are timelike then there are two cases; one is that both the surface S_1 and the curve $\alpha_1(s_1)$ fully lying on S_1 are timelike. In this case we say that the pair $\{\alpha, \alpha_1\}$ is a Mannheim D-pair of the type 3. The other case is that the curve $\alpha_1(s_1)$ fully lying on S_1 is a spacelike curve. Then the pair $\{\alpha, \alpha_1\}$ is a Mannheim D-pair of the type 4. If the curve $\alpha(s)$ lying on S is a spacelike curve then both the surface S_1 and the curve $\alpha_1(s_1)$ fully lying on S_1 are spacelike. Then we say that the pair $\{\alpha, \alpha_1\}$ is a Mannheim D-pair of the type 5 [9].

Definition 2.3. *Let S be an oriented surface in Minkowski 3-space E_1^3 with unit normal Z. For any constant r in IR, let S_r is given by $S_r = \{f(p) = p + rZ_p : \ p \in S\}$. Thus if p is on S, then $f(p) = p + rZ_p$ defines a new surface S_r. The map f is called the natural map on S into S_r, and if f is univalent, then S_r is a parallel surface of S with unit normal Z, such that $Z_{f(p)} = Z_p$ for all p on S.*

3. Bertrand Partner D-Curves on Parallel Surfaces in E_1^3

In this section, by considering parallel surfaces we deal with the notion of Bertrand partner D-curves in E_1^3. We will consider the Bertrand D-pair of the type 3. The results of the other types can be obtained by using the same procedures of this section. So, here in after when we talk about the pair $\{\alpha, \alpha_1\}$ we will mean that $\{\alpha, \alpha_1\}$ is a Bertrand D-pair of the type 3 and the pair (S, S_r) is a parallel timelike surface pair in E_1^3.

Theorem 3.1. *Let $\{\alpha, \alpha_1\}$ be a Bertrand D-pair of the type 3 and the pair (S, S_r) be a parallel timelike surface pair in E_1^3. The curve $\beta(s_\beta)$ lying fully on S_r be the image of the curve $\alpha(s)$ lying fully on S. Then the relationship between the Darboux frames of $\alpha(s)$ and $\beta(s_\beta)$ is given as follows*

$$\begin{bmatrix} T^* \\ Y^* \\ Z^* \end{bmatrix} = \begin{bmatrix} \frac{1+rk_n}{\sqrt{|(rt_r)^2-(1+rk_n)^2|}} & \frac{rt_r}{\sqrt{|(rt_r)^2-(1+rk_n)^2|}} & 0 \\ \pm\frac{rt_r}{\sqrt{|(rt_r)^2-(1+rk_n)^2|}} & \pm\frac{1+rk_n}{\sqrt{|(rt_r)^2-(1+rk_n)^2|}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix}. \tag{3}$$

Proof. Let S and S_1 be oriented timelike surfaces in Minkowski 3-space E_1^3 and let consider arc-length parameter Bertrand partner D-curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1, respectively. Denote the Darboux frames and invariants of $\alpha(s)$ and $\alpha_1(s_1)$ by $\{T, Y, Z\}$, k_g, k_n, t_r and $\{T_1, Y_1, Z_1\}$, k_{g_1}, k_{n_1}, t_{r_1}, respectively. Then from the definition of Bertrand partner D-curves we have

$$\alpha_1(s_1) = \alpha(s_1) - \lambda Y_1(s_1). \tag{4}$$

(See [10]). For the oriented timelike surfaces S_r and S_{r_1} assume that surface pairs (S, S_r) and (S_1, S_{r_1}) are parallel surfaces in E_1^3. Then from (4) the images of the curves $\alpha(s)$ and $\alpha_1(s_1)$ on the surfaces S_r and S_{r_1} are given by

$$\beta(s_\beta) = \alpha(s) + rZ(s), \tag{5}$$

$$\beta_1(s_{\beta_1}) = \alpha(s_1) - \lambda Y_1(s_1) + r_1 Z_1(s_1), \tag{6}$$

respectively. Denote the Darboux frames of $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ by $\{T^*, Y^*, Z^*\}$ and $\{T_1^*, Y_1^*, Z_1^*\}$, respectively. Differentiating (5) with respect to s we have

$$\frac{d\beta}{ds} = \frac{d\beta}{ds_\beta}\frac{ds_\beta}{ds} = \alpha'(s) + rZ', \tag{7}$$

here s_β is the arc length parameter of the curve $\beta(s_\beta)$. By considering Darboux derivative formulae, from (7) it follows

$$T^*\frac{ds_\beta}{ds} = (1 + rk_n)T + rt_r Y, \tag{8}$$

which gives us

$$\frac{ds_\beta}{ds} = \sqrt{\left|(rt_r)^2 - (1 + rk_n)^2\right|}. \tag{9}$$

From (8) and (9) we have

$$T^{*}=\frac{1}{\sqrt{\left|(rt_r)^2-(1+rk_n)^2\right|}}\left[(1+rk_n)\,T+rt_rY\right]. \tag{10}$$

The curve $\beta(s_\beta)$ can be timelike or spacelike. Then we have $Y^{*}=\pm Z\times T^{*}$ and from (10) it is obtained that

$$Y^{*}=\frac{\pm1}{\sqrt{\left|(rt_r)^2-(1+rk_n)^2\right|}}\left[rt_rT+(1+rk_n)\,Y\right]. \tag{11}$$

Then from (10) and (11) we obtain the matrix form given in (3). □

Theorem 3.2. *Let the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1 respectively, be Bertrand Partner D-curves of the type 3 and the pair (S_1,S_{r_1}) be a parallel surface pair. If the curve $\beta_1(s_{\beta_1})$ lying fully on S_{r_1} is the image of the curve $\alpha_1(s_1)$ under the natural mapping f, then the relationship between the Darboux frames of $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ is given as follows,*

(12)

$$\begin{bmatrix} T_1^{*}\\ Y_1^{*}\\ Z_1^{*}\end{bmatrix}=\begin{bmatrix} \frac{1+r_1k_{n_1}}{\sqrt{\left|(r_1t_{r_1})^2-(1+r_1k_{n_1})^2\right|}} & \frac{r_1t_{r_1}}{\sqrt{\left|(r_1t_{r_1})^2-(1+r_1k_{n_1})^2\right|}} & 0\\ \frac{r_1t_{r_1}}{\sqrt{\left|(r_1t_{r_1})^2-(1+r_1k_{n_1})^2\right|}} & \frac{(1+r_1k_{n_1})}{\sqrt{\left|(r_1t_{r_1})^2-(1+r_1k_{n_1})^2\right|}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} T_1\\ Y_1\\ Z_1\end{bmatrix}.$$

Proof. From the differentiation of (6) with respect to s_1 it follows

$$\frac{d\beta_1}{ds_1}=\frac{d\beta_1}{ds_{\beta_1}}\frac{ds_{\beta_1}}{ds_1}=\frac{d\alpha}{ds}\frac{ds}{ds_1}-\lambda Y_1'+r_1Z_1'. \tag{13}$$

Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves of the type 3 we have

$$T\frac{ds}{ds_1}=(1+\lambda k_{g_1})\,T_1-\lambda t_{r_1}Z_1, \tag{14}$$

(See [9]). Then substituting (14) in (13) it follows

$$T_1^{*}\frac{ds_{\beta_1}}{ds_1}=(1+r_1k_{n_1})\,T_1+r_1t_{r_1}Y_1, \tag{15}$$

which gives us

$$\frac{ds_{\beta_1}}{ds_1}=\sqrt{\left|(r_1t_{r_1})^2-(1+r_1k_{n_1})^2\right|}. \tag{16}$$

Thus (15) becomes

$$T_1^{*}=\frac{1}{\sqrt{\left|(r_1t_{r_1})^2-(1+r_1k_{n_1})^2\right|}}\left[T_1\,(1+r_1k_{n_1})+r_1t_{r_1}Y_1\right]. \tag{17}$$

Here $\beta_1(s_{\beta_1})$ can be timelike or spacelike. Thus, we have $Y_1^{*}=\pm Z_1\times T_1^{*}$ and from (17) we get

$$Y_1^{*}=\pm\frac{1}{\sqrt{\left|(r_1t_{r_1})^2-(1+r_1k_{n_1})^2\right|}}\left[r_1t_{r_1}T_1+(1+r_1k_{n_1})\,Y_1\right]. \tag{18}$$

Then from (17) and (18) we have the matrix form given in (12). □

Moreover, since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves we have

$$\frac{ds}{ds_1} = -\frac{1+\lambda k_{g_1}}{\lambda t_{r_1}}. \tag{19}$$

Then from (9), (16) and (19) we have the following corollary:

Corrolary 3.1. *Let the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1 respectively, be Bertrand Partner D-curves and the pairs (S, S_r) and (S_1, S_{r_1}) be parallel surface pairs. Then the relationship between arc length parameters s_{β_1} and s_β is given by*

$$s_{\beta_1} = -\int \sqrt{\frac{\left|(r_1 t_{r_1})^2 - (1 + r_1 k_{n_1})^2\right|}{\left|(r t_r)^2 - (1 + r k_n)^2\right|}}\, \frac{\lambda t_{r_1}}{1+\lambda k_{g_1}} ds_\beta. \tag{20}$$

After these computations, we can give the following characterizations. Here in after we assume that the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on surfaces S and S_1 respectively, are Bertrand Partner D-curves of the type 3, the pairs (S, S_r) and (S_1, S_{r_1}) are parallel surface pairs, the curves $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are images of the curves $\alpha(s)$ and $\alpha_1(s_1)$ on S_r and S_{r1}, respectively.

Theorem 3.3. *$\alpha(s)$ is a principal line on S if and only if $\alpha(s)$ and $\beta(s_\beta)$ are Bertrand partner D-curves.*

Proof. Let $\alpha(s)$ be a principal line on S. Then we have $t_r = 0$ and from (11) it follows $Y^* = \pm Y$ i.e., $\alpha(s)$ and $\beta(s_\beta)$ are Bertrand D-curves.

Conversely, if $\alpha(s)$ and $\beta(s_\beta)$ are Bertrand D-curves, then from (11) we have $t_r = 0$, i.e., $\alpha(s)$ is a principal line on S. □

Theorem 3.4. *$\alpha_1(s_1)$ is a principal line on S_1 if and only if $\alpha(s)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.*

Proof. Let $\alpha_1(s_1)$ be a principal line on S_1 i.e., $t_{r_1} = 0$. Then from (18) we have $Y_1^* = \pm Y$. It means that $\alpha(s)$ and $\beta_1(s_{\beta_1})$ are Bertrand D-curves in E_1^3.

Conversely, if $\alpha(s)$ and $\beta_1(s_{\beta_1})$ are Bertrand D-curves, then from (18) we have $t_{r_1} = 0$, i.e., $\alpha_1(s_1)$ is a principal line on S_1. □

Theorem 3.5. *$\alpha(s)$ is a principal line on S if and only if $\alpha_1(s_1)$ and $\beta(s_\beta)$ are Bertrand partner D-curves.*

Proof. Let $\alpha(s)$ be a principal line on S. Then we have $t_r = 0$ and from (11) we have $Y^* = \pm Y$. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves we have $Y = Y_1$. Then we obtain $Y^* = \pm Y_1$, i.e., $\alpha_1(s_1)$ and $\beta(s_\beta)$ are Bertrand partner D-curves.

Conversely, if $\alpha_1(s_1)$ and $\beta(s_\beta)$ are Bertrand partner D-curves, by considering the condition that $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves, from (11) it is obtained that $t_r = 0$. Then $\alpha(s)$ is a principal line on S. □

Theorem 3.6. *$\alpha_1(s_1)$ is a principal line on S_1 if and only if $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.*

Proof. If $\alpha_1(s_1)$ is a principal line on S_1, i.e., $t_{r_1} = 0$, then (18) gives us $Y_1^* = \pm Y_1$. Then $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.

Conversely, if $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves, then from (18) we have $t_{r_1} = 0$, i.e., $\alpha_1(s_1)$ is a principal line on S_1. □

Theorem 3.7. *$\alpha(s)$ and $\alpha_1(s_1)$ are principal lines on S and S_1, respectively if and only if $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.*

Proof. If $\alpha(s)$ and $\alpha_1(s_1)$ are principal lines on S and S_1, respectively, then $t_r = 0$, $t_{r_1} = 0$. In this case from (11) and (18) we have $Y^* = \pm Y$ and $Y_1^* = \pm Y_1$, respectively. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves, $Y = Y_1$. From the last two equalities we obtain that $Y^* = \pm Y_1^*$ which means that $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.

Let now $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ be Bertrand partner D-curves. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves, from (11) we have

$$Y_1 = \pm \frac{1}{1 + rk_n} \left[\left(\sqrt{\left| (rt_r)^2 - (1 + rk_n)^2 \right|} \right) Y^* \mp rt_r T \right] \tag{21}$$

Substituting (21) in (18) gives

$$Y_1^* = \pm \frac{1}{\sqrt{\left| (r_1 t_{r_1})^2 - (1 + r_1 k_{n_1})^2 \right|}} \left[r_1 t_{r_1} T_1 \pm \left(\sqrt{\left| (rt_r)^2 - (1 + rk_n)^2 \right|} \right) Y^* \mp rt_r T \right] \tag{22}$$

Since we assume $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves, from (22) it is obtained that $t_r = 0$, $t_{r_1} = 0$, i.e., $\alpha(s)$ and $\alpha_1(s_1)$ are principal lines on S and S_1, respectively. □

Theorem 3.8. *$\alpha(s)$ is both geodesic curve and principal line on S if and only if $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.*

Proof. Let $\alpha(s)$ be both geodesic curve and principal line on S, i.e., $k_g = t_r = 0$. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves we have

$$t_{r1} = (-k_g \sinh\theta + t_r \cosh\theta) \frac{ds}{ds_1} \tag{23}$$

(See [9]). Under the condition $k_g = t_r = 0$, from (23) and (24) we have $t_{r_1} = k_{g_1} = 0$, i.e., $\alpha_1(s_1)$ is also a geodesic and a principal line on S_1. Then from Theorem 3.7, we have that $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ Bertrand partner D-curves.

Conversely, if $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves, from Theorem 3.7, $\alpha(s)$ and $\alpha_1(s_1)$ are principal lines on S and S_1, respectively, i.e., $t_r = 0$, $t_{r_1} = 0$. Then for the non-trivial case $\theta \neq 0$, from (23) we have $k_g = 0$. Then $\alpha(s)$ is both geodesic curve and principal line on S. □

Theorem 3.9. *If $\alpha(s)$ is both geodesic curve and principal line on S then $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.*

Proof. Let $\alpha(s)$ be both geodesic curve and principal line on S. Then from (23) and (24) we have $t_{r_1} = k_{g_1} = 0$. Then from (18) we have $Y_1^* = \pm Y_1$, i.e., $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand D-curves. □

Theorem 3.10. *If $\alpha(s)$ is both geodesic curve and principal line on S then $\alpha(s)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.*

Proof. Let $\alpha(s)$ be both geodesic curve and principal line on S. Then using (23) and (24) from (18) we have $Y_1^* = \pm Y_1$. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand patner D-curves, $Y = Y_1$, and so we have $Y_1^* = \pm Y$. □

4. Mannheim Partner D-Curves on Parallel Surfaces in E_1^3

In this section, we deal with the notion of Mannheim partner D-curves by considering parallel surfaces in E_1^3. Similar to the Bertrand case, we will consider the Mannheim partner D-pairs of the type 3. The results of the other types can be obtained by the same way of this section. So, when we talk about the curve pair $\{\alpha, \alpha_1\}$ we will mean that $\{\alpha, \alpha_1\}$ is a Mannheim partner D-pair of the type 3 in E_1^3.

Theorem 4.1. *Let the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1 respectively, be Mannheim partner D-curves of the type 3 and the pair (S_1, S_{r_1}) be a parallel surface pair in E_1^3. If the curve $\beta_1(s_{\beta_1})$ lying fully on S_{r_1} is the image of $\alpha_1(s_1)$ under the natural mapping f, then the relationships between the Darboux frames of $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are given as follows,*

(24)
$$\begin{bmatrix} T_1^* \\ Y_1^* \\ Z_1^* \end{bmatrix} = \begin{bmatrix} \frac{1+r_1k_{n_1}}{\sqrt{\left|(r_1t_{r_1})^2-(1+r_1k_{n_1})^2\right|}} & \frac{r_1t_{r_1}}{\sqrt{\left|(r_1t_{r_1})^2-(1+r_1k_{n_1})^2\right|}} & 0 \\ \pm\frac{r_1t_{r_1}}{\sqrt{\left|(r_1t_{r_1})^2-(1+r_1k_{n_1})^2\right|}} & \pm\frac{1+r_1k_{n_1}}{\sqrt{\left|(r_1t_{r_1})^2-(1+r_1k_{n_1})^2\right|}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ Y_1 \\ Z_1 \end{bmatrix}.$$

Proof. Let S and S_1 be oriented timelike surfaces in 3-dimensional Minkowski space E_1^3 and let consider the arc-length parameter Mannheim partner D-curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1, respectively. Denote the Darboux frames and invariants of $\alpha(s)$ and $\alpha_1(s_1)$ by $\{T, Y, Z\}$, k_g, k_n, t_r and $\{T_1, Y_1, Z_1\}$, k_{g_1}, k_{n_1}, t_{r_1}, respectively. Then from the definition of Mannheim partner D-curves we have

$$\alpha_1(s_1) = \alpha(s_1) - \lambda Z_1(s_1), \tag{25}$$

(See [9]). For the oriented timelike surfaces S_r and S_{r_1} assume that surface pairs (S, S_r) and (S_1, S_{r_1}) are parallel surfaces. Then from (22) the images of the curves $\alpha(s)$ and $\alpha_1(s_1)$ on the surfaces S_r and S_{r_1} are given by

$$\beta(s_\beta) = \alpha(s) + rZ(s), \tag{26}$$

$$\beta_1(s_{\beta_1}) = \alpha(s_1) + (r_1 - \lambda)Z_1(s_1) \tag{27}$$

respectively. Denote the Darboux frames of $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ by $\{T^*, Y^*, Z^*\}$ and $\{T_1^*, Y_1^*, Z_1^*\}$, respectively. Differentiating (27) with respect to s_1 we have

$$\frac{d\beta_1}{ds_1} = \frac{d\beta_1}{ds_{\beta_1}}\frac{ds_{\beta_1}}{ds_1} = \frac{d\alpha(s_1)}{ds}\frac{ds}{ds_1} + (r_1 - \lambda)Z_1'(s_1). \tag{28}$$

Similarly, differentiating (25) with respect to s_1 it follows

$$T\frac{ds}{ds_1} = (1+\lambda k_{n_1})\,T_1 + \lambda t_{r_1} Y_1. \tag{29}$$

Then substituting (29) in (28) it follows

$$T_1^*\frac{ds_{\beta_1}}{ds_1} = (1+r_1 k_{n_1})\,T_1 + (r_1 t_{r_1})\,Y_1, \tag{30}$$

which gives us

$$\frac{ds_{\beta_1}}{ds_1} = \sqrt{\left|(r_1 t_{r_1})^2 - (1+r_1 k_{n_1})^2\right|}. \tag{31}$$

Thus (30) becomes

$$T_1^* = \frac{1}{\sqrt{\left|(r_1 t_{r_1})^2 - (1+r_1 k_{n_1})^2\right|}}\left[(1+r_1 k_{n_1})\,T_1 + (r_1 t_{r_1})\,Y_1\right] . \tag{32}$$

In this case, $\beta_1\left(s_{\beta_1}\right)$ can be timelike or spacelike. Then we have $Y_1^* = \pm Z_1 \times T_1^*$ and from (32) we have

$$Y_1^* = \pm\frac{1}{\sqrt{\left|(r_1 t_{r_1})^2 - (1+r_1 k_{n_1})^2\right|}}\left[(r_1 t_{r_1})\,T_1 + (1+r_1 k_{n_1})\,Y_1\right]. \tag{33}$$

Then from (32) and (33) we have the matrix form given in (24). Moreover, since $\alpha(s)$ and $\alpha_1(s_1)$ are Mannheim partner D-curves we have

$$\frac{ds}{ds_1} = \frac{1+\lambda k_{n_1}}{\lambda t_{r_1}}, \tag{34}$$

(See [9]). □

Then from (9), (31) and (34) we have the following corollary.

Corrolary 4.1. *Let the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1 respectively, be Mannheim Partner D-curves of the type 3 and the pairs (S,S_r) and (S_1,S_{r_1}) be parallel surface pairs. Then the relationship between arc length parameters s_{β_1} and s_β is given by*

$$s_{\beta_1} = \int\sqrt{\frac{\left|(r_1 t_{r_1})^2 - (1+r_1 k_{n_1})^2\right|}{\left|(r t_r)^2 - (1+r k_n)^2\right|}}\,\frac{\lambda t_{r_1}}{\lambda k_{n_1}+1} ds_\beta \tag{35}$$

After these computations, we can give the following characterizations. Here in after we assume that the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on surfaces S and S_1 respectively, are Mannheim Partner D-curves of the type 3, the pairs (S,S_r) and (S_1,S_{r_1}) are parallel surface pairs in E_1^3, the curves $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are images of the curves $\alpha(s)$ and $\alpha_1(s_1)$ on S_r and S_{r_1}, respectively.

Theorem 4.2. *$\alpha_1(s_1)$ is a principal line on S_1 if and only if $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves in E_1^3.*

Proof. Let $\alpha_1(s_1)$ be a principal line on S_1. Then from (33) we have $Y_1^* = \pm Y_1$, i.e., $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.

Conversely, if $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves, then from (33) we have that $\alpha_1(s_1)$ is a principal line on S_1. □

Theorem 4.3. *$\alpha(s)$ and $\beta_1(s_{\beta_1})$ are Mannheim partner D-curves if and only if* $\tanh\theta = \frac{r_1 t_{r_1}}{1+r_1 k_{n_1}}$ *holds, where θ is the angle between unit tangents T_1 and T.*

Proof. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Mannheim partner D-curves of the type 3, we have $T_1 = \cosh\theta T - \sinh\theta Z$, $Y_1 = -\sinh\theta T + \cosh\theta Z$. Then from (33) we obtain

$$\begin{aligned} Y_1^* = \pm \frac{1}{\sqrt{\left|\left(r_1 t_{r_1}\right)^2 - \left(1+r_1 k_{n_1}\right)^2\right|}} & \left[\left(r_1 t_{r_1}\cosh\theta - (1+r_1 k_{n_1})\sinh\theta\right) T \right. \\ & \left. + \left(-r_1 t_{r_1}\sinh\theta + (1+r_1 k_{n_1})\cosh\theta\right) Z\right] \end{aligned} \tag{36}$$

From (36) it is clear that $\alpha(s)$ and $\beta_1(s_{\beta_1})$ are Mannheim partner D-curves if and only if $\tanh\theta = \frac{r_1 t_{r_1}}{1+r_1 k_{n_1}}$ holds. □

Corrolary 4.2. *$\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are Mannheim partner D-curves if and only if* $\tanh\theta = \frac{r_1 t_{r_1}}{1+r_1 k_{n_1}}$ *holds, where θ is the angle between unit tangents T_1 and T.*

Proof. Since $\alpha(s)$ and its image curve $\beta(s_\beta)$ have same unit normal direction Z, from Theorem 4.3, we obtain that $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are Mannheim partner D-curves if and only if $\tanh\theta = \frac{r_1 t_{r_1}}{1+r_1 k_{n_1}}$ holds. □

Corrolary 4.3. *$\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves if and only if* $\tanh\theta = \frac{r_1 t_{r_1}}{1+r_1 k_{n_1}}$ *holds, where θ is the angle between unit tangents T_1 and T.*

Proof. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Mannheim partner D-curves, we have $Z = \pm Y_1$. Then from (36) it is clear that $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves if and only if $\tanh\theta = \frac{r_1 t_{r_1}}{1+r_1 k_{n_1}}$ holds. □

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