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ON SOME WEIGHTED ERDÖS-MORDELL'S TYPE INEQUALITIES FOR POLYGONS

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Abstract. In this paper, we establish a simple proof of the Wolstenholme cyclic inequality and generalized main result of the Bombardelli and Wu [1].

1. Introduction

Let $\triangle A_1A_2A_3$ be a triangle, and let P be an interior point of $\triangle A_1A_2A_3$. We denote the distance from P to the vertices by $PA_i = r_i, i = 1, 2, 3$, and the distances from P to the sides A_1A_2, A_2A_3, A_3A_1, by $d_{1,2}, d_{2,3}, d_{3,1}$, respectively. The famous Erdös-Mordell inequality asserts that

$$r_1 + r_2 + r_3 \geq 2(d_{1,2} + d_{2,3} + d_{3,1}) \tag{1}$$

with equality if and only if the triangle is equilateral and the point P is its center.

The (1) inequality was proposed by Erdös [4] as a conjectured and proved by Mordell and Barrow [11]. Some related results with historical comments on this problem can be found in [2, 8, 10].

Let $\mathcal{P}_n$ be a convex polygon with $n \geq 3$ vertices, and let P be an interior point of $\mathcal{P}_n$. The distances from P to the sides $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1$ are denoted, respectively, by $d_{1,2}, d_{2,3}, \ldots, d_{n-1,n}, d_{n,1}$ and $w_{i,i+1}$ denote the length of the bisector of the angle A_iPA_{i+1} from P to its intersection with the side A_iA_{i+1} $(i = 1, 2, \ldots, n, A_{n+1} = A_1)$. The distances PA_i are denoted by $r_i, i = 1, 2, \ldots, n$. We also denote $\theta_{i,i+1} = \angle A_iPA_{i+1}, i = 1, 2, \ldots, n$, where the index i is taken modulo n.

Lenhard [9] established a remarkable inequality concerning the convex polygon as an extension of Erdös-Mordell inequality as follows

$$\sum_{i=1}^{n} r_i \geq \frac{1}{\cos\frac{\pi}{n}} \sum_{i=1}^{n} w_{i,i+1} \geq \frac{1}{\cos\frac{\pi}{n}} \sum_{i=1}^{n} d_{i,i+1}. \tag{2}$$

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Dar and Gueron [3] proved a weighted Erdös-Mordell inequality:

$$\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 \geq 2\left(\sqrt{\lambda_1\lambda_2}d_{1,2} + \sqrt{\lambda_2\lambda_3}d_{2,3} + \sqrt{\lambda_3\lambda_1}d_{3,1}\right) \tag{3}$$

where $\lambda_1, \lambda_2, \lambda_3$ are positive numbers.

In a recent paper, Gueron and Shafrir [5] generalized (3) as follows:

$$\sum_{i=1}^{n} \lambda_i r_i \geq \left(\sec\frac{\pi}{n}\right)\sum_{i=1}^{n}\sqrt{\lambda_i\lambda_{i+1}}d_{i,i+1}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive numbers and $\lambda_{n+1} = \lambda_1$.

In 2004, Janous [7] generalized Dar-Gueron's inequality by introducing an exponential parameter, as follows

$$\lambda_1 r_1^t + \lambda_2 r_2^t + \lambda_3 r_3^t \geq 2^{\min\{t,1\}}\left(\sqrt{\lambda_1\lambda_2}d_{1,2}^t + \sqrt{\lambda_2\lambda_3}d_{2,3}^t + \sqrt{\lambda_3\lambda_1}d_{3,1}^t\right)$$

where $\lambda_1, \lambda_2, \lambda_3$ and t are positive real numbers.

In a recent paper [13], Shanhe Wu sharpened Janous's inequality in the following form:

$$\lambda_1 r_1^t + \lambda_2 r_2^t + \lambda_3 r_3^t \geq 2^{\min\{t,1\}}\left(\sqrt{\lambda_1\lambda_2}\omega_{1,2}^t + \sqrt{\lambda_2\lambda_3}\omega_{2,3}^t + \sqrt{\lambda_3\lambda_1}\omega_{3,1}^t\right)$$

where $\lambda_1, \lambda_2, \lambda_3$ and t are positive real numbers.

2. Lemmas

Lemma 2.1. *For any positive $x_1, x_2, \ldots, x_n$ we have,*

$$\sum_{i=1}^{n} x_i^2 \geq \frac{1}{\cos\frac{\pi}{n}} \cdot \sum_{i=1}^{n} x_i x_{i+1}\cos\left(\frac{\theta_{i,i+1}}{2}\right).$$

Proof. In a coordinate plane we choose the points $B_i(u_i, v_i), i = \overline{1,n}$ such that $OB_i = x_i, \angle B_iOB_{i+1} = \frac{\theta_{i,i+1}}{2}, i = \overline{1,n}$, where the index i is taken modulo n. Let $B_iB_{i+1} = b_i, i = \overline{1,n}$. Apply the cosine theorem for each of the triangles $B_iOB_{i+1}, i = \overline{1,n}$ we have

$$b_i^2 = x_i^2 + x_{i+1}^2 - 2x_ix_{i+1}\cos\frac{\theta_{i,i+1}}{2}.$$

It follows that,

$$\begin{aligned} x_ix_{i+1}\cos\frac{\theta_{i,i+1}}{2} &= \frac{1}{2}(u_i^2 + v_i^2 + u_{i+1}^2 + v_{i+1}^2 - (u_{i+1} - u_i)^2 - (v_{i+1} - v_i)^2) \\ &= u_iu_{i+1} + v_iv_{i+1},\ (\forall i \in \{1, 2, \ldots, n-1\}) \end{aligned}$$

and

$$\begin{aligned} x_1x_n\cos\frac{\theta_{i,i+1}}{2} &= x_1x_n\cos\left(\pi - \sum_{i=1}^{n-1}\frac{\theta_{i,i+1}}{2}\right) \\ &= -x_1x_n\cos\left(\sum_{i=1}^{n-1}\frac{\theta_{i,i+1}}{2}\right) \\ &= \frac{1}{2}(x_1^2 + x_n^2 - b_n^2) = -u_1u_n - v_1v_n. \end{aligned}$$

Therefore,

$$\cos\frac{\pi}{n}\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}x_ix_{i+1}\cos\frac{\theta_{i,i+1}}{2}=$$

$$\cos\pi n\cdot\sum_{i=1}^{n}(u_i^2+v_i^2)-\sum_{i=1}^{n-1}(u_iu_{i+1}+v_iv_{i+1})+u_1u_n+v_1v_n.$$

It is suffices to prove following two inequality,

(4) $$u_1u_2+\cdots+u_{n-1}u_n-u_1u_n\le\cos\frac{\pi}{n}(u_1^2+u_2^2+\cdots+u_n^2),$$

(5) $$v_1v_2+\cdots+v_{n-1}v_n-v_1v_n\le\cos\frac{\pi}{n}(v_1^2+u_2^2+\cdots+v_n^2).$$

Now we will prove the inequality (4). By the same method the inequality (5) will be proved. First we observed that for any positive real numbers x_1, y_1 following inequality holds

$$\left(\sqrt{x_1}u_1-\frac{1}{2\sqrt{x_1}}u_2+\sqrt{y_1}u_n\right)^2\ge 0$$

i.e.

$$u_1u_2+\sqrt{\frac{y_1}{x_1}}u_2u_n-2\sqrt{x_1y_1}u_1u_n\le x_1u_1^2+\frac{1}{4x_1}u_2^2+y_1u_n^2.$$

By the same way for any positive real numbers $x_2, y_2, \dots, x_{n-2}, y_{n-2}$ we get that

$$u_2u_3+\sqrt{\frac{y_2}{x_2}}u_3u_n-2\sqrt{x_2y_2}u_2u_n\le x_2u_2^2+\frac{1}{4x_2}u_3^2+y_2u_n^2$$

$$\cdots\ \cdots\ \cdots$$

$$u_{n-3}u_{n-2}+\sqrt{\frac{y_{n-3}}{x_{n-3}}}u_{n-2}u_n-2\sqrt{x_{n-3}y_{n-3}}u_{n-3}u_n\le x_{n-3}u_{n-3}^2+\frac{1}{4x_{n-3}}u_{n-2}^2+y_{n-3}u_n^2$$

(6)

$$u_{n-2}u_{n-1}+\sqrt{\frac{y_{n-2}}{x_{n-2}}}u_{n-1}u_n-2\sqrt{x_{n-2}y_{n-2}}u_{n-2}u_n\le x_{n-2}u_{n-2}^2+\frac{1}{4x_{n-2}}u_{n-1}^2+y_{n-2}u_n^2.$$

We choose the numbers $x_1, x_2, \dots, x_{n-2}, y_1, y_2, \dots, y_{n-2}$ such that

$$x_1y_1=\frac{1}{4},\ \sqrt{\frac{y_1}{x_1}}=2\sqrt{x_2y_2},\dots,\sqrt{\frac{y_{n-3}}{x_{n-3}}}=2\sqrt{x_{n-2}y_{n-2}},\ y_{n-2}=x_{n-2}$$

and

(7) $$x_1=\frac{1}{4x_1}+x_2=\cdots=\frac{1}{4x_{n-3}}+x_{n-2}=\frac{1}{4x_{n-2}}=y_1+y_2+\cdots+y_{n-2}.$$

It is easy to see that $0<x_1<1$. Thus there exists $\alpha\in(0;\frac{\pi}{2})$ such that $x_1=\cos\alpha$. From the recurrent equation of (7) we find the numbers $x_2, \dots, x_{n-2}$ as follows

$$x_2=\frac{\sin 3\alpha}{2\sin 2\alpha},\dots,x_{n-2}=\frac{\sin(n-1)\alpha}{2\sin(n-2)\alpha}.$$

Hence

$$x_1=\frac{1}{4x_{n-2}}\Leftrightarrow\cos\alpha=\frac{\sin(n-2)\alpha}{2\sin(n-1)\alpha}.$$

From here we get that $\alpha = \frac{\pi}{n}$. Now we find the numbers $y_1, y_2, \dots, y_{n-2}$:

$$\begin{aligned} y_1 &= \frac{1}{4x_1} = \frac{1}{4\cos\frac{\pi}{n}} = \frac{\sin^2\frac{\pi}{n}}{2\sin\frac{\pi}{n}\sin\frac{2\pi}{n}}, \\ y_2 &= \frac{y_1}{4x_1x_2} = \frac{1}{4}\cdot\frac{1}{4\cos^2\frac{\pi}{n}}\cdot\frac{2\sin\frac{2\pi}{n}}{\sin\frac{3\pi}{n}} = \frac{\sin^2\frac{\pi}{n}}{2\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}}, \\ &\dots \quad \dots\ \dots \\ y_{n-2} &= \frac{\sin(n-1)\frac{\pi}{n}}{2\sin(n-2)\frac{\pi}{n}} = \frac{\sin^2\frac{\pi}{n}}{2\sin\frac{(n-2)\pi}{n}\cdot\sin\frac{(n-1)\pi}{n}}. \end{aligned}$$

Finally, summing all inequality in (6) we have

$$\begin{aligned} u_1u_2 + u_2u_3 + \cdots + u_{n-1}u_n - u_1u_n &\le x_1\cdot(u_1^2 + u_2^2 + \cdots + u_n^2) \\ &= \cos\frac{\pi}{n}(u_1^2 + u_2^2 + \cdots + u_n^2) \end{aligned}$$

then (4) inequality is proved. □

Lemma 2.2 (see [6, 14]). *If $a_i > 0, \lambda_i > 0, i = 1, 2, \dots, n$, $0 < t \le 1$ then*

$$\sum_{i=1}^{n} a_i^t \le \left(\sum_{i=1}^{n}\lambda_i\right)^{1-t}\cdot\left(\sum_{i=1}^{n}\lambda_i a_i\right)^t.$$

Lemma 2.3. *Let $x_i > 0, 0 < \varphi_i < \frac{\pi}{2} (i = 1, 2, \dots, n), n \ge 3$ and $\sum_{i=1}^n \varphi_i = \pi$. Then for $t > 0$ we have,*

$$\sum_{i=1}^{n} x_ix_{i+1}\cos^t\varphi_i \le \left(\cos\frac{\pi}{n}\right)^{\min\{t,1\}}\cdot\sum_{i=1}^{n}x_i^2.$$

Proof. *Case*(I). Let $0 < t \le 1$. Using Lemma 2.2 we get,

$$\begin{aligned} \sum_{i=1}^{n} x_ix_{i+1}\cos^t\varphi_i &\le \left(\sum_{i=1}^{n}x_ix_{i+1}\right)^{1-t}\cdot\left(\sum_{i=1}^{n}x_ix_{i+1}\cos\varphi_i\right)^t \\ &\qquad \text{(Applying Cauchy inequality together with Lemma 2.1)} \\ &\le \left(x_1^2 + \cdots + x_n^2\right)^{1-t}\left(\cos\frac{\pi}{n}\sum_{i=1}^{n}x_i^2\right)^t \\ &= \left(\cos\frac{\pi}{n}\right)^t\cdot\sum_{i=1}^{n}x_i^2. \end{aligned}$$

Case(II). Let $t > 1$. Using Lemma 2.1 we get,

$$\begin{aligned} \sum_{i=1}^{n} x_ix_{i+1}\cos^t\varphi_i &< \sum_{i=1}^{n}x_ix_{i+1}\cos\varphi_i \\ &\le \cos\frac{\pi}{n}\sum_{i=1}^{n}x_i^2. \end{aligned}$$

Lemma 2.3 is proved. □

Lemma 2.4 (see [1]). *Let Q be an interior point of polygon $A_1A_2\ldots A_n$, and let $\angle A_1QA_2 = 2\alpha_1, \angle A_2QA_3 = 2\alpha_2, \ldots, \angle A_nQA_1 = 2\alpha_n$. The bisectors of $\angle A_1QA_2, \angle A_2QA_3, \ldots, \angle A_nQA_1$ intersect respectively the circumcircles of $\triangle A_1QA_2, \triangle A_2QA_3, \ldots, \triangle A_nQA_1$ in the points $A_1', A_2', \ldots, A_n'$. Let $QA_i = r_i, QA_i' = l_{i,i+1}, i = 1, 2, \ldots, n$. Then, we have the following identities*

$$l_{1,2} = \frac{r_1 + r_2}{2\cos\alpha_1}, l_{2,3} = \frac{r_2 + r_3}{2\cos\alpha_2}, \ldots, l_{n,1} = \frac{r_n + r_1}{2\cos\alpha_n}.$$

3. Main results

Theorem 3.1. *Suppose Q is an interior point of polygon $A_1A_2\ldots A_n$ and conditions of Lemma 2.4 holds. Then for $\lambda_i > 0 (i = 1, 2, \ldots, n)$ and $t < 0$, we have the inequality*

$$\left(\cos\frac{\pi}{n}\right)^{\min\{-t,1\}} \cdot \sum_{i=1}^{n} \lambda_i r_i^t \geq \sum_{i=1}^{n} \sqrt{\lambda_i\lambda_{i+1}} \cdot l_{i,i+1}^t. \tag{8}$$

Proof. From Lemma 2.4 that

$$\sum_{i=1}^{n} \sqrt{\lambda_i\lambda_{i+1}} \cdot l_{i,i+1}^t = \sum_{i=1}^{n} \sqrt{\lambda_i\lambda_{i+1}} \left(\frac{2\cos\alpha_i}{r_i + r_{i+1}}\right)^{-t}. \tag{9}$$

Since $-t > 0$, by using the arithmetic-geometric means inequality and the inequality given by Lemma 2.3, we obtain:

$$\begin{aligned} \sum_{i=1}^{n} \sqrt{\lambda_i\lambda_{i+1}} \left(\frac{2\cos\alpha_i}{r_i + r_{i+1}}\right)^{-t} &\leq \sum_{i=1}^{n} \sqrt{\lambda_i\lambda_{i+1} r_i^t r_{i+1}^t} (\cos^{-t}\alpha_i) \\ &\leq \left(\cos\frac{\pi}{n}\right)^{\min\{-t,1\}} \cdot \sum_{i=1}^{n} \lambda_i r_i^t. \end{aligned} \tag{10}$$

Combining identity (9) and inequality (10) yields the inequality (8). □

4. Application to Erdös-Mordell's type inequalities of Lemma 2.1

Theorem 4.1 (see [5]). *For any positive $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have,*

$$\sum_{i=1}^{n} \lambda_i r_i \geq \frac{1}{\cos\frac{\pi}{n}} \cdot \sum_{i=1}^{n} \sqrt{\lambda_i\lambda_{i+1}} d_{i,i+1}.$$

Proof. Choose the numbers such that $x_i = \sqrt{\lambda_i r_i}$, $i = \overline{1,n}$. From the Lemma 2.1 we have,

$$\sum_{i=1}^{n} \lambda_i r_i \geq \frac{1}{\cos\frac{\pi}{n}} \sum_{i=1}^{n} \sqrt{\lambda_i\lambda_{i+1}} \sqrt{r_i r_{i+1}} \cos\frac{\theta_{i,i+1}}{2}$$

and then apply the inequality

$$\sqrt{r_i r_{i+1}} \cos \frac{\theta_{i,i+1}}{2} \geq \omega_{i,i+1} \geq d_{i,i+1}, \quad \left(i = \overline{1,n}\right)$$

we have,

$$\sum_{i=1}^{n} \lambda_i r_i \geq \frac{1}{\cos \frac{\pi}{n}} \cdot \sum_{i=1}^{n} \sqrt{r_i r_{i+1}} d_{i,i+1}.$$

The proof of Theorem 4.1 is complete. □

Theorem 4.2 (see [13]). *For any positive* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *we have,*

$$\sum_{i=1}^{n} \lambda_i r_i^t \geq \left(\cos \frac{\pi}{n}\right)^{-\min\{t,1\}} \cdot \sum_{i=1}^{n} \sqrt{\lambda_i \lambda_{i+1}} \cdot \omega_{i,i+1}^t.$$

Proof. Choose the numbers $x_i, i = \overline{1,n}$ such that $x_i = \sqrt{\lambda_i r_i^t}, i = \overline{1,n}$. From the Lemma 2.3 we have,

$$\left(\cos \frac{\pi}{n}\right)^{\min\{t,1\}} \cdot \sum_{i=1}^{n} \lambda_i r_i^t$$

$$\geq \sum_{i=1}^{n} \sqrt{\lambda_i \lambda_{i+1}} \cdot \left(\sqrt{r_i r_{i+1}} \cos \varphi_i\right)^t$$

and then apply the inequality

$$\sqrt{r_i r_{i+1}} \cos \varphi_i \geq \omega_{i,i+1} \geq d_{i,i+1}, \quad \left(i = \overline{1,n}\right)$$

we get,

$$\left(\cos \frac{\pi}{n}\right)^{\min\{t,1\}} \sum_{i=1}^{n} \lambda_i r_i^t \geq \sum_{i=1}^{n} \sqrt{\lambda_i \lambda_{i+1}} \cdot \omega_{i,i+1}^t.$$

This completes the proof of Theorem 4.2. □

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