A GENERALIZATION OF THE ISOGONAL POINT

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Abstract. In this paper we give a generalization of the isogonal point in terms of concurrency, starting with the idea of "breaking" the equilateral triangles constructed on the sides of one triangle, rotating them with the same acute angle and compressing with the same ratio the broken sides.

1. Introduction

We will denote by $\mathcal{E}_2$ the euclidean space.

Definition 1.1. For $X \in \mathcal{E}_2$ fixed and $\alpha \in (-\pi; \pi)$, we call rotation of center $X$ and angle $\alpha$ of $Y \in \mathcal{E}_2$, the point $z = R_{x,\alpha}(Y)$, having the following properties:

(i) $m(\angle YXZ) = \alpha$;
(ii) $\overline{XY} = \overline{XZ}$.

Definition 1.2. Let $O \in \mathcal{E}_2$ a fixed point and $k \in \mathbb{R}\setminus\{0\}$. We call homothety with center $O$ and ratio $k$ an application:

$$H_{0,k} : \mathcal{E}_2 \rightarrow \mathcal{E}_2 : H_{0,k}(M) = M'$$

with the following properties:

(1) $H_{0,k}(O) = O$;
(2) If $M \neq 0$, then the points $O, M, M'$ are collinear;
(3) If $k > 0$, then $M' \in [OM]$, and if $k < 0$, then $O \in [MM']$;
(4) $OM' = |k| \cdot OM$.

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2. Main Results

If we consider an arbitrary triangle $ABC$ and the points $C_a = (H_{A,k} \circ R_{A,-\alpha})(B)$, $C_b = (H_{B,k} \circ R_{B,k})(A)$, $A_b = (H_{B,k} \circ R_{B,k})(C)$, $A_c = (H_{C,k} \circ R_{C,-\alpha})(B)$, $B_c = (H_{C,k} \circ R_{C,-\alpha})(A)$, $B_a = (H_{A,k} \circ R_{A,a})(C)$ with $k \in \mathbb{R}^*$ and $\alpha \in (-\pi; \pi)$ fixed.

**Lemma 2.1.** If $C', B'$ are points in the interior of the triangle $ABC$ with $\triangle ABC' \sim \triangle ACB'$ and $CC' \cap AB = \{C_1\}$, also $BB' \cap AC = \{B_1\}$, then the following identity is true:

$$\frac{BA'}{AC} = \frac{BC' \cdot \sin(C + y)}{AB \cdot \sin(A + x)},$$

with $y \equiv m(\angle ABC') = m(\angle ACB')$.

**Proof.** We have

$$\frac{B_1A}{B_1C} = \frac{S_{\triangle BB'C}}{S_{\triangle BAB'}} = \frac{BC' \cdot BC' \cdot \sin(C + y)}{AB \cdot AB' \cdot \sin(A + x)} = \frac{\sin C \cdot \sin y \cdot \sin(A + x)}{\sin A \cdot \sin x \cdot \sin(C + y)},$$

where $x \equiv m(\angle BAC') = m(\angle CAB')$ (see Figure 1).

![Figure 1](attachment:image.png)

Similarly, we have

$$\frac{C_1A}{C_1B} = \frac{S_{\triangle ACC'}}{S_{\triangle CCB'}} = \frac{AC' \cdot AC \cdot \sin(A + x)}{BC' \cdot BC \cdot \sin(B + y)} = \frac{\sin y \cdot \sin B \cdot \sin(A + x)}{\sin x \cdot \sin A \cdot \sin(B + y)}.$$

Now, using the Ceva’s Theorem, in the triangle $ABC$, one obtains

$$\frac{A'C}{BA'} = \frac{B_1C \cdot C_1A}{B_1A \cdot C_1B} = \frac{\sin B \cdot \sin(C + y)}{\sin C \cdot \sin(B + y)}.$$

**Theorem 2.1.** Let us consider now an arbitrary triangle $ABC$ and the points $C_a = (H_{A,k} \circ R_{A,-\alpha})(B)$, $C_b = (H_{B,k} \circ R_{B,k})(A)$, $A_b = (H_{B,k} \circ R_{B,k})(C)$, $A_c = (H_{C,k} \circ R_{C,-\alpha})(B)$, $B_c = (H_{C,k} \circ R_{C,-\alpha})(A)$, $B_a = (H_{A,k} \circ R_{A,a})(C)$, with $k \in \mathbb{R}^*$, $\alpha \in (-\pi; \pi)$ fixed. Using this points, defined above, we consider the following intersections: $BB_c \cap CC_b = \{P_a\}$, $AP_a \cap BC = \{P_A\}$, $BB_a \cap AA_b = \{P_c\}$, $CP_c \cap AB = \{P_C\}$, $CC_a \cap AA_c = \{P_b\}$, $BB_b \cap
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Let $AC = \{P_B\}$. The concurrency of the cevians $AP_A, BP_B, CP_C$ takes place in the point $T_{\alpha,k}$.

**Proof.** Applying Lemma 2.1 one obtains:

\[
\begin{align*}
BP_A &= \sin C \cdot \sin(B + \alpha) \\
PA_C &= \sin B \cdot \sin(C + \alpha) \\
CP_B &= \sin A \cdot \sin(C + \alpha) \\
PB_A &= \sin C \cdot \sin(A + \alpha) \\
AP_C &= \sin B \cdot \sin(A + \alpha) \\
P_B &= \sin A \cdot \sin(B + \alpha).
\end{align*}
\]

Now, evaluating the product

\[
BP_A \cdot CP_B \cdot AP_C = \frac{1}{P_A \cdot PB \cdot PC}
\]

and from the reciprocal of Ceva’s Theorem, we have the concurrency in $T_{\alpha,k}$ (see Figure 2).

**Corollary 2.1.** For $\alpha = 60^\circ$ one obtains the first Torricelli point, and for $\alpha = -60^\circ$ one obtains the second Torricelli point, replacing $k = 1$.

**Corollary 2.2.** For $k = 1/(2 \cos \alpha)$, one obtains: $C_a = C_b; B_a = B_c; A_c = A_b$ and the $ABC_a$, $BA_bC$, $CB_aA$ are similar isosceles triangles and the Theorem of Kiepert takes place.[1].

**Corollary 2.3.** From Lemma 2.2 one obtaines that the geometrical locus of the intersection point $CC' \cap BB'$ is the line $AA'$, where

\[
BA'/A'C = \frac{\sin C \cdot \sin(B + y)}{\sin B \cdot \sin(C + y)}.
\]
Theorem 2.2. Let us consider now an arbitrary triangle $ABC$ and the following points $C_a = (H_{A,k} \circ R_{A,-\alpha})(B)$, $C_b = (H_{B,k} \circ R_{B,\alpha})(A)$, $A_b = (H_{B,k} \circ R_{B,\alpha})(C), A_c = (H_{C,k} \circ R_{C,-\alpha})(B)$, $B_c = (H_{C,k} \circ R_{C,-\alpha})(A)$, $B_a = (H_{A,k} \circ R_{A,\alpha})(C)$ with $k \in \mathbb{R}^*$, $\alpha \in (-\pi; \pi)$ fixed. Using this points, we consider the following intersections: $Q_a = CC_a \cap BB_a$, $Q_b = AA_b \cap CC_b$, $Q_c = BB_c \cap AA_c$, $AQ_a \cap BC = \{Q_A\}$, $BQ_b \cap AC = \{Q_B\}$, $CQ_c \cap AB = \{Q_C\}$. The concurrency of lines $AQ_A, BQ_B, CQ_C$ takes place in the point $T_{\alpha,k}^*$. 

Proof. We make the following notations: $\beta = m(\angle C_A BA) = m(\angle C_B AB) = m(\angle B_A CA) = m(\angle B_C AC) = m(\angle A_C BC) = m(\angle A_B CB)$.

Applying Lemma 2.1 one obtains:

$$
\begin{align*}
BQ_A &= \frac{\sin C \cdot \sin(B + \beta)}{\sin B \cdot \sin(C + \beta)} \\
Q_A C &= \sin B \cdot \sin(C + \beta) \\
CQ_B &= \sin A \cdot \sin(C + \beta) \\
Q_B A &= \sin C \cdot \sin(A + \beta) \\
A Q_C &= \sin B \cdot \sin(A + \beta) \\
Q_C B &= \sin A \cdot \sin(B + \beta)
\end{align*}
$$

Now, applying again the reciprocal of the Ceva’s Theorem, one obtains the required concurrency in the point $T_{\alpha,k}^*$ (See Figure 3).

Remark 2.1. The corollary 2.1, 2.2 and 2.3 takes place also for Theorem 2.2.

Remark 2.2. The barycentric coordinates for the points $T_{k,\alpha}$ and $T_{\alpha,k}^*$ can remain an open problem.

Remark 2.3. For $k = 1$, Theorem 2.1 become Theorem 1.1 from [4].
REFERENCES


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