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A GENERALIZATION OF THE ISOGONAL POINT

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Abstract. In this paper we give a generalization of the isogonal point in terms of concurrency, starting with the ideea of "breaking" the equilateral triangles constructed on the sides of one triangle, rotating them with the same acute angle and compressing with the same ratio the broken sides.

1. Introduction

We will denote by $\mathcal{E}_2$ the euclidean space.

Definition 1.1. For $X \in \mathcal{E}_2$ fixed and $\alpha \in (-\pi;\pi)$, we call *rotation of center X and angle* α of $Y \in \mathcal{E}_2$, the point $z = R_{x,\alpha}(Y)$, having the following properties:

(i) $m(\measuredangle YXZ) = \alpha$;
(ii) $\overrightarrow{XY} = \overrightarrow{XZ}$.

Definition 1.2. Let $O \in \mathcal{E}_2$ a fixed point and $k \in \mathbb{R}\backslash\{0\}$. We call *homothety with center O and ratio k* an application:

$$H_{0,k} : \mathcal{E}_2 \longrightarrow \mathcal{E}_2 : H_{0,k}(M) = M'$$

with the following properties:

(1) $H_{0,k}(O) = O$;
(2) If $M \neq 0$, then the points O, M, M' are collinear;
(3) If $k > 0$, then $M' \in [OM$, and if $k < 0$, then $O \in [MM']$;
(4) $OM' = |k| \cdot OM$.

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2. Main Results

If we consider an arbitrary triangle ABC and the points $C_a = (H_{A,k} \circ R_{A,-\alpha})(B)$, $C_b = (H_{B,k} \circ R_{B,\alpha})(A)$, $A_b = (H_{B,k} \circ R_{B,\alpha})(C)$, $A_c = (H_{C,k} \circ R_{C,-\alpha})(B)$, $B_c = (H_{C,k} \circ R_{C,-\alpha})(A)$, $B_a = (H_{A,k} \circ R_{A,\alpha})(C)$ with $k \in \mathbb{R}^*$ and $\alpha \in (-\pi;\pi)$ fixed.

Lemma 2.1. *If C', B' are points in the interior of the triangle ABC with $\triangle ABC' \sim \triangle ACB'$ and $CC' \cap AB = \{C_1\}$, also $BB' \cap AC = \{B_1\}$, then the following identity is true:*

$$\frac{BA'}{A'C} = \frac{\sin C \cdot \sin(B+y)}{\sin B \cdot \sin(C+y)},$$

with $y \overset{not}{=} m(\measuredangle ABC') = m(\measuredangle ACB')$.

Proof. We have

$$\frac{B_1A}{B_1C} = \frac{S_{\triangle BB'C}}{S_{\triangle BAB'}} = \frac{BC \cdot B'C \cdot \sin(C+y)}{AB \cdot AB' \cdot \sin(A+x)} = \frac{\sin C \cdot \sin y \cdot \sin(A+x)}{\sin A \cdot \sin x \cdot \sin(C+y)},$$

where $x \overset{not}{=} m(\measuredangle BAC') = m(\measuredangle CAB')$ (see Figure 1).

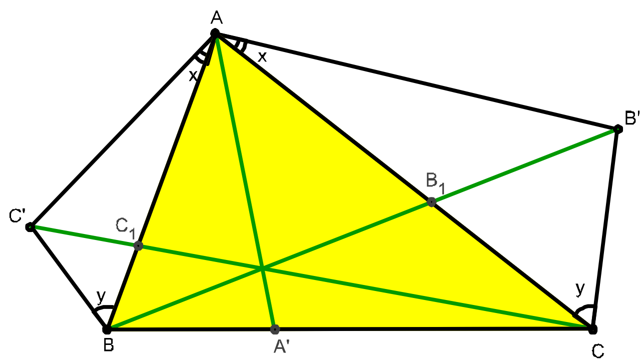

Figure 1

Similarly, we have

$$\frac{C_1A}{C_1B} = \frac{S_{\triangle ACC'}}{S_{\triangle CC'B}} = \frac{AC' \cdot AC \cdot \sin(A+x)}{BC' \cdot BC \cdot \sin(B+y)} = \frac{\sin y \cdot \sin B \cdot \sin(A+x)}{\sin x \cdot \sin A \cdot \sin(B+y)}.$$

Now, using the Ceva's Theorem, in the triangle ABC, one obtains

$$\frac{A'C}{BA'} = \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = \frac{\sin B \cdot \sin(C+y)}{\sin C \cdot \sin(B+y)}.$$

□

Theorem 2.1. *Let us consider now an arbitrary triangle ABC and the points $C_a = (H_{A,k} \circ R_{A,-\alpha})(B)$, $C_b = (H_{B,k} \circ R_{B,k})(A)$, $A_b = (H_{B,k} \circ R_{B,\alpha})(C)$, $A_c = (H_{C,k} \circ R_{C,-\alpha})(B)$, $B_c = (H_{C,k} \circ R_{C,-\alpha})(A)$, $B_a = (H_{A,k} \circ R_{A,\alpha})(C)$, with $k \in \mathbb{R}^*$, $\alpha \in (-\pi;\pi)$ fixed. Using this points, defined above, we consider the following intersections: $BB_c \cap CC_b = \{P_a\}$, $AP_a \cap BC = \{P_A\}$, $BB_a \cap AA_b = \{P_c\}$, $CP_c \cap AB = \{P_C\}$, $CC_a \cap AA_c = \{P_b\}$, $BP_b \cap$*

$AC = \{P_B\}$. The concurrency of the cevians AP_A, BP_B, CP_C takes place in the point $T_{\alpha,k}$.

Proof. Applying Lema 2.1 one obtains:

$$\frac{BP_A}{P_AC} = \frac{\sin C \cdot \sin(B+\alpha)}{\sin B \cdot \sin(C+\alpha)}$$
$$\frac{CP_B}{P_BA} = \frac{\sin A \cdot \sin(C+\alpha)}{\sin C \cdot \sin(A+\alpha)}$$
$$\frac{AP_C}{P_CB} = \frac{\sin B \cdot \sin(A+\alpha)}{\sin A \cdot \sin(B+\alpha)}.$$

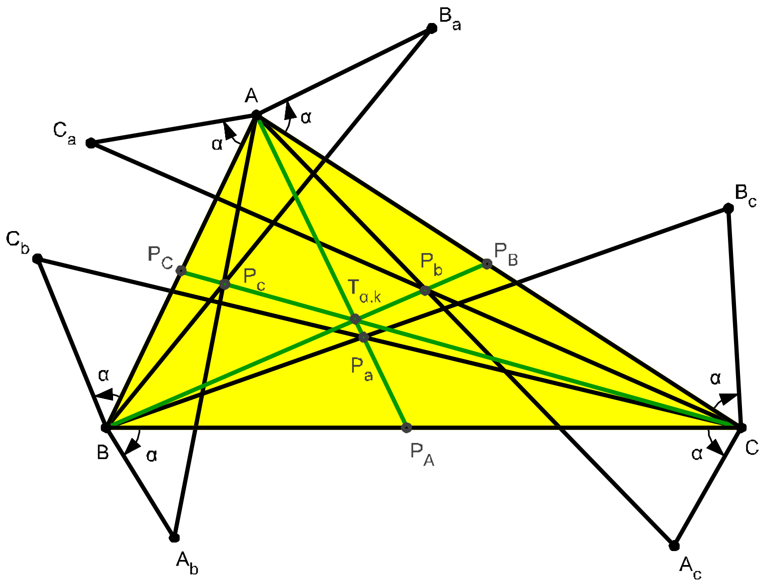

Figure 2

Now, evaluating the product

$$\frac{BP_A}{P_AC} \cdot \frac{CP_B}{P_BA} \cdot \frac{AP_C}{P_CB} = 1$$

and from the reciprocal of Ceva's Theorem, we have the concurrency in $T_{\alpha,k}$ (see Figure 2). □

Corrolary 2.1. *For $\alpha = 60^\circ$ one obtains the first Torricelli point, and for $\alpha = -60^\circ$ one obtains the second Torricelli point, replacing $k = 1$.*

Corrolary 2.2. *For $k = 1/(2\cos\alpha)$, one obtains: $C_a = C_b$; $B_a = B_c$; $A_c = A_b$ and the ABC_a, BA_bC, CB_aA are similar isosceles triangles and the Theorem of Kiepert takes place.*[1].

Corrolary 2.3. *From Lemma 2.2 one obtaines that the geometrical locus of the intersection point $CC' \cap BB'$ is the line AA', where*

$$BA'/A'C = \frac{\sin C \cdot \sin(B+y)}{\sin B \cdot \sin(C+y)}.$$

Theorem 2.2. *Let us consider now an arbitrary triangle ABC and the following points $C_a = (H_{A,k} \circ R_{A,-\alpha})(B)$, $C_b = (H_{B,k} \circ R_{B,\alpha})(A)$, $A_b = (H_{B,k} \circ R_{B,\alpha})(C)$,$A_c = (H_{C,k} \circ R_{C,-\alpha})(B)$, $B_c = (H_{C,k} \circ R_{C,-\alpha})(A)$, $B_a = (H_{A,k} \circ R_{A,\alpha})(C)$ with $k \in \mathbb{R}^*$, $\alpha \in (-\pi;\pi)$ fixed. Using this points, we consider the following intersections: $\{Q_a\} = CC_a \cap BB_a$, $\{Q_b\} = AA_b \cap CC_b$, $\{Q_c\} = BB_c \cap AA_c$, $AQ_a \cap BC = \{Q_A\}$, $BQ_b \cap AC = \{Q_B\}$, $CQ_c \cap AB = \{Q_C\}$. The concurrency of lines AQ_A, BQ_B, CQ_C takes place in the point $T^*_{\alpha,k}$.*

Proof. We make the following notations: $\beta = m(\measuredangle C_aBA) = m(\measuredangle C_bAB) = m(\measuredangle B_aCA) = m(\measuredangle B_cAC) = m(\measuredangle A_cBC) = m(\measuredangle A_bCB)$.

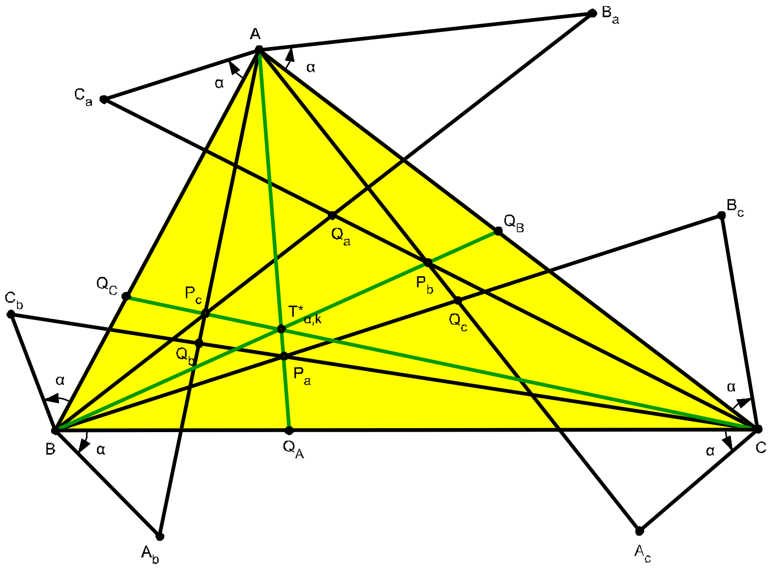

Figure 3

Applying Lemma 2.1 one obtains:

$$\frac{BQ_A}{Q_AC} = \frac{\sin C \cdot \sin(B+\beta)}{\sin B \cdot \sin(C+\beta)}$$
$$\frac{CQ_B}{Q_BA} = \frac{\sin A \cdot \sin(C+\beta)}{\sin C \cdot \sin(A+\beta)}$$
$$\frac{AQ_C}{Q_CB} = \frac{\sin B \cdot \sin(A+\beta)}{\sin A \cdot \sin(B+\beta)}.$$

Now, applying again the reciprocal of the Ceva's Theorem, one obtains the required concurrency in the point $T^*_{\alpha,k}$ (See Figure 3). □

Remark 2.1. The corrolary 2.1, 2.2 and 2.3 takes place also for Theorem 2.2.

Remark 2.2. The barycentric coordinates for the points $T_{k,\alpha}$ and $T^*_{k,\alpha}$ can remain an open problem.

Remark 2.3. For $k = 1$, Theorem 2.1 become Theorem 1.1 from [4].

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