Abstract. In this paper we present the several geometric inequalities of Erdös-Mordell type in the convex polygon, using the Cauchy Inequality.

1. Introduction

In [6], in colaboration with A. Gobej, we present some geometric inequalities of Erdös-Mordell type in the convex polygon. Here, we found others geometric inequalities of Erdös-Mordell type, using several known inequalities, in the convex polygon.

Let $A_1, A_2, \ldots, A_n$ the vertices of the convex polygon, $n \geq 3$, and $M$, a point interior to the polygon. We note with $R_k$ the distances from $M$ to the vertices $A_k$ and we note with $r_k$ the distances from $M$ to the sides $[A_kA_{k+1}]$ of length $A_kA_{k+1} = a_k$, where $k = \frac{1}{2}n$ and $A_{n+1} \equiv A_1$. For all $k \in \{1, \ldots, n\}$ with $A_{n+1} \equiv A_1$ and $m(A_kMA_{k+1}) = \delta_k$ we have the following property:

$$\delta_1 + \delta_2 + \ldots + \delta_n = 2\pi.$$

L. Fejes Tóth conjectured a inequality which is refered to the convex polygon, recall in [1] și [3], thus

$$\sum_{k=1}^{n} r_k \leq \cos \left( \frac{\pi}{n} \right) \sum_{k=1}^{n} R_k.$$

In 1961 H.-C. Lenhard proof the inequality (1), used the inequality

$$\sum_{k=1}^{n} w_k \leq \cos \left( \frac{\pi}{n} \right) \sum_{k=1}^{n} R_k,$$

which was established in [5], where $w_k$ the length of the bisector of the angle $A_kMA_{k+1}$, $(\forall) k = \frac{1}{2}n$ with $A_{n+1} \equiv A_1$.

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M. Dincă published other solution for inequality (1) in Gazeta Matematică Seria B in 1998 (see [4]). Another inequality of Erdős-Mordell type for convex polygon was given by N. Ozeki [9] in 1957, namely,\n\n\[ \prod_{k=1}^{n} R_k \geq \left( \frac{\pi}{n} \right)^n \prod_{k=1}^{n} w_k \]

which proved the inequality 16.8 from [3] due to L. Fejes-Tóth, so \n\n\[ \prod_{k=1}^{n} R_k \geq \left( \frac{\pi}{n} \right)^n \prod_{k=1}^{n} r_k. \]

R. R. Janić in [3], shows that in any convex polygon \( A_1A_2...A_n \), there is the inequality \n\n\[ \sum_{k=1}^{n} R_k \sin \frac{A_k}{2} \geq \sum_{k=1}^{n} r_k. \]

D. Buşneag proposed in GMB no. 1/1971 the problem 10876, which is an inequality of Erdős-Mordell type for convex polygon, thus, \n\n\[ \sum_{k=1}^{n} \frac{a_k}{r_k} \geq \frac{2p^2}{\Delta}, \]

where \( p \) is the semiperimeter of polygon \( A_1A_2...A_n \) and \( \Delta \) is the area of polygon.

In connection with inequality (6), D. M. Bătineţu established [2] the inequality \n\n\[ \sum_{k=1}^{n} \frac{a_k}{r_k} \geq \frac{2p}{r} \]

if the polygon \( A_1A_2...A_n \) is circumscribed about a circle of radius \( r \).

Among the relations established between the elements of polygon \( A_1A_2...A_n \), we can remark the following relation for \( \Delta \)- the area of convex polygon \( A_1A_2...A_n \):

\[ 2\Delta = a_1r_1 + a_2r_2 + ... + a_nr_n. \]

We select several inequalities obtained from [6]: \n\n\[ R_k \geq \frac{r_{k-1} + r_k}{2 \sin \frac{A_k}{2}} \]

hold for all \( k \in \{1, 2, ..., n\} \), with \( r_0 = r_n \), \n\n\[ \left( 2 \cos \frac{\pi}{n} \right)^n \prod_{k=1}^{n} R_k \geq \prod_{k=1}^{n} (r_{k-1} + r_k), \quad (r_0 = r_n) \]

and \n\n\[ \sum_{k=1}^{n} \frac{r_{k-1} + r_k}{R_k} \leq 2n \cos \frac{\pi}{n} \]

and \n\n\[ \sum_{k=1}^{n} \frac{R_k^2}{r_k} \geq \sec \frac{\pi}{n} \sum_{k=1}^{n} R_k. \]
2. Main results

First, we will follow some procedures used in paper [6], through which we will obtain some Erdős-Mordell-type inequalities for the convex polygon. Among these will apply the Cauchy Inequality

**Theorem 2.1.** In any convex polygon $A_1A_2\ldots A_n$, there is the inequality

\[
\sum_{k=1}^{n} r_k \frac{r_k}{R_k + R_{k+1}} \leq 2n \cos \frac{\pi}{n}.
\]

**Proof.** The inequality (11),

\[
\sum_{k=1}^{n} \frac{r_{k-1} + r_k}{R_k} \leq 2n \cos \frac{\pi}{n},
\]

with $r_0 = r_n$, is expanded in the following way,

\[
r_n + r_1 \frac{1}{R_1} + r_1 + r_2 \frac{1}{R_2} + \ldots + \frac{r_{n-1} + r_n}{R_n} = r_1 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + r_2 \left( \frac{1}{R_2} + \frac{1}{R_3} \right) + \ldots + r_n \left( \frac{1}{R_n} + \frac{1}{R_1} \right) \leq 2n \cos \frac{\pi}{n}
\]

On the other hand, we have

\[
\frac{1}{R_{k-1}} + \frac{1}{R_k} \geq \frac{4}{R_{k-1} + R_k}, \quad (\forall) \ k = 1, \ldots, n
\]

with $R_0 = R_n$, from where we can deduce another inequality, of an Erdős-Mordell type, namely,

\[
\sum_{k=1}^{n} \frac{r_k}{R_k + R_{k+1}} \leq \frac{n}{2} \cos \frac{\pi}{n}.
\]

$\square$

**Theorem 2.2.** For any convex polygon $A_1A_2\ldots A_n$, we have the inequality

\[
\sum_{k=1}^{n} \frac{R_k + R_{k+1}}{r_k} \geq \frac{2n}{\cos \frac{\pi}{2}},
\]

with $R_{n+1} = R_1$.

**Proof.** The inequality

\[
\sum_{k=1}^{n} x_k y_k \cdot \sum_{k=1}^{n} \frac{x_k}{y_k} \geq \left( \sum_{k=1}^{n} x_k \right)^2
\]

is well known, because it is a particulary case of Cauchy’s inequality. In this way we will take $x_k = \frac{r_k}{R_k + R_{k+1}}$ and $y_k = 1$. Thus, the inequality becomes

\[
\sum_{k=1}^{n} \frac{r_k}{R_k + R_{k+1}} \cdot \sum_{k=1}^{n} \frac{R_k + R_{k+1}}{r_k} \geq n^2
\]
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and, if we use the inequality (13), we get

\[
\sum_{k=1}^{n} \frac{R_k + R_{k+1}}{r_k} \geq \frac{2n}{\cos \frac{n}{2}}
\]

with \(R_{n+1} = R_1\).

**Theorem 2.3.** In any convex polygon \(A_1A_2..A_n\), there is the inequality

(15)

\[
\sum_{k=1}^{n} \frac{\sqrt{r_{k-1}r_k}}{R_k} \leq n \cos \frac{\pi}{n}.
\]

**Proof.** From Cauchy’s inequality, we have

\[
\sum_{k=1}^{n} x_k y_k \cdot \sum_{k=1}^{n} \frac{x_k}{y_k} \geq \left( \sum_{k=1}^{n} x_k \right)^2.
\]

Using the substitutions

\[
x_k = \frac{\sqrt{r_{k-1}r_k}}{R_k}
\]

and \(y_k = 1\), we deduce that the inequality

\[
\sum_{k=1}^{n} \frac{\sqrt{r_{k-1}r_k}}{R_k} \cdot \sum_{k=1}^{n} \frac{R_k}{\sqrt{r_{k-1}r_k}} \geq n^2,
\]

holds. However, from the relation (11), we obtain

\[
\sum_{k=1}^{n} \frac{\sqrt{r_{k-1}r_k}}{R_k} \leq n \cos \frac{\pi}{n}
\]

which implies the inequality

\[
\sum_{k=1}^{n} \frac{R_k}{\sqrt{r_{k-1}r_k}} \geq \frac{n}{\cos \frac{\pi}{n}},
\]

with \(r_0 = r_n\).

**Remark 1.** The inequality (15) generalizes the problem 1045 of G. Tsinsifas from the magazine *Crux Mathematicorum*. This is also remarked in [7].

**Theorem 2.4.** In any convex polygon \(A_1A_2..A_n\) there is the inequality

(16)

\[
\sum_{k=1}^{n} \frac{R_k^2}{\sqrt{r_{k-1}r_k}} \geq \frac{n^2}{\cos^2 \frac{\pi}{n}},
\]

with \(r_0 = r_n\).

**Proof.** In the inequality

\[
\sum_{k=1}^{n} x_k y_k \cdot \sum_{k=1}^{n} \frac{x_k}{y_k} \geq \left( \sum_{k=1}^{n} x_k \right)^2
\]

we replaced \(x_k = y_k = \frac{R_k}{\sqrt{r_{k-1}r_k}}\) and the inequality becomes

\[
n \sum_{k=1}^{n} \frac{R_k^2}{\sqrt{r_{k-1}r_k}} \geq \left( \sum_{k=1}^{n} \frac{R_k}{\sqrt{r_{k-1}r_k}} \right)^2 \geq \frac{n^2}{\cos^2 \frac{\pi}{n}}.
\]
This means that inequality (16) is true.

**Theorem 2.5.** In any convex polygon $A_1A_2...A_n$ there is the inequality

$$\sum_{k=1}^{n} \frac{a_k r_k}{R_k R_{k+1}} \leq n \frac{2\pi}{n},$$

with $A_{n+1} = A_1$.

**Proof.** We can be written the area of triangle $A_kMA_{k+1}$ in two ways, thus

$$\frac{a_k r_k}{2} = R_k R_{k+1} \sin A_k M A_{k+1},$$

which implies the relation

$$\frac{a_k r_k}{R_k R_{k+1}} = \sin A_k M A_{k+1}$$

so, by passing to the sum, we get the relation

$$\sum_{k=1}^{n} \frac{a_k r_k}{R_k R_{k+1}} = \sum_{k=1}^{n} \sin A_k M A_{k+1},$$

with $A_{n+1} = A_1$. Because the function $f : (0, \infty) \to R$, is defined as $f(x) = \sin x$, is concave, we will apply the inequality Jensen, thus

$$\frac{1}{n} \sum_{k=1}^{n} \sin A_k M A_{k+1} \leq \sin \left( \frac{\sum_{k=1}^{n} A_k M A_{k+1}}{n} \right) = \sin \frac{2\pi}{n}.$$ 

Therefore, we have

$$\sum_{k=1}^{n} A_k M A_{k+1} \leq n \sin \frac{2\pi}{n}.$$ 

Consequently, we obtain the inequality of the statement. 

**Remark 2.** The equality hold in the above mentioned theorems when the polygon is regular.

**Remark 3.** On the a hand, we have the equality (8),

$$2\Delta = a_1 r_1 + a_2 r_2 + ... + a_n r_n = \sum_{k=1}^{n} a_k r_k,$$

and on the other hand, we have the inequality Cauchy, where we will replace $x_k = \sqrt{a_k r_k}$ and $y_k = \frac{a_k}{r_k}$, then

$$2\Delta \sum_{k=1}^{n} \frac{a_k}{r_k} = \sum_{k=1}^{n} a_k r_k \sum_{k=1}^{n} \frac{a_k}{r_k} \geq \left( \sum_{k=1}^{n} a_k \right)^2 = 4p^2$$

which proves the inequality (6).

**Theorem 2.6.** In any convex polygon $A_1A_2...A_n$ there is the inequality

$$\sum_{k=1}^{n} R_k^2 \sin \frac{A_k}{2} \geq \frac{\sec \frac{\pi}{n}}{n} \left( \sum_{k=1}^{n} r_k \right)^2$$

holds.
Proof. Applying inequality (9), we have the inequality
\[ R_k = MA_k \geq \frac{r_{k-1} + r_k}{2 \sin \frac{A_k}{2}} \quad (\forall) \quad k = 1, n \]
with \( r_0 = r_n \), and this, by squaring, becomes
\[ 4R_k^2 \sin \frac{A_k}{2} \geq \frac{(r_{k-1} + r_k)^2}{\sin \frac{A_k}{2}} \]
and taking the sum, we deduce
\[ \sum_{k=1}^{n} R_k^2 \sin \frac{A_k}{2} \geq \frac{4 \left( \sum_{k=1}^{n} r_k \right)^2}{\sum_{k=1}^{n} \sin \frac{A_k}{2}} \]
so, we found inequality (18).

\[ \square \]

**Theorem 2.7.** In any convex polygon \( A_1A_2...A_n \) there is the inequality
\[ \sum_{k=1}^{n} (R_k + R_{k+1}) r_k \geq \frac{2 \sec \frac{n}{2}}{n} \left( \sum_{k=1}^{n} r_k \right)^2 \]
holds.

Proof. From inequality (9), we have
\[ R_k = MA_k \geq \frac{r_{k-1} + r_k}{2 \sin \frac{A_k}{2}}, \quad (\forall) \quad k = 1, n \]
with \( r_0 = r_n \), and this, by multiply with \((r_{k-1} + r_k)\), becomes
\[ 2 (r_{k-1} + r_k) R_k \geq \frac{(r_{k-1} + r_k)^2}{\sin \frac{A_k}{2}} \]
and by passing to the sum, we obtain the relation
\[ 2 \sum_{k=1}^{n} (r_{k-1} + r_k) R_k \geq \sum_{k=1}^{n} \frac{(r_{k-1} + r_k)^2}{\sin \frac{A_k}{2}} \geq \frac{4 \left( \sum_{k=1}^{n} r_k \right)^2}{\sum_{k=1}^{n} \sin \frac{A_k}{2}}. \]
Therefore, we have
\[ \sum_{k=1}^{n} (r_{k-1} + r_k) R_k \geq \frac{2 \sec \frac{n}{2}}{n} \left( \sum_{k=1}^{n} r_k \right)^2. \]
But, it follows that
\[ \sum_{k=1}^{n} (R_k + R_{k+1}) r_k = \sum_{k=1}^{n} (r_{k-1} + r_k) R_k \]
which means that, we obtain the inequality of statement. \( \square \)
References


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