TWO NEW PROOFS OF
GOORMAGHTIGH’S THEOREM

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Abstract. In this note we present two new demonstrations of the theorem of a Belgian mathematician René Goormaghtigh.

1. Introduction

In order to state our main results we need recall some important theorems that we need in proving the Goormaghtigh’s theorem. Consider a triangle $ABC$ is neither isosceles rectangular nor with circumcenter $O$. We present below an interesting proposition given by Goormaghtigh.

**Theorem 1.1.** (Goormaghtigh [7, pp. 281 – 283]). Let $A', B', C'$ be points on $OA, OB, OC$ so that

$$
\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = k,
$$

$k \in \mathbb{R}^*_+$, then the intersections of the perpendiculars to $OA$ at $A'$, $OB$ at $B'$, and $OC$ at $C'$ with the respective sidelines $BC, CA, AB$ are collinear.

R. Musselman and R. Goormaghtigh are given in [7] a proof of this theorem using complex numbers. A synthetic demonstration is also given by K. Nguyen meet in [9].

**Theorem 1.2.** (Kariya [3, p. 109]). Let $C_a, C_b, C_c$ the points of tangency of the incircle with the sides $BC, CA, AB$ of triangle $ABC$ and $I$ center of the incircle. On the lines $IC_a, IC_b, IC_c$ the points $A', B', C''$ are considered in the same direction so that $IA' = IB' = IC''$. Then the lines $AA', BB', CC'$ are concurrent.

**Theorem 1.3.** (Desargues [5, p. 133]). Two triangles are in axial perspective if and only if they are in central perspective.

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Theorem 1.4. (Miquel [6, pp. 233–234]). The centers of the circles of the four triangles of a complete quadrilateral are on a circle. (Miquel’s Circle).

Theorem 1.5. (Steiner [6, p. 235]). Miquel’s point of the circles determined by the four triangles of a complete quadrilateral is situated on Miquel’s circle.

Theorem 1.6. (Sondat [11, p. 10]). If two triangles $ABC$ and $A'B'C'$ are perspective and orthologic, then the center of perspective $P$ and the orthologic centers $Q$ and $Q'$ are on the same line perpendicular to the axis of perspectivity $d$.

Theorem 1.7. (Thébault [12, pp. 22–24]). If two triangles $ABC$ and $A'B'C'$ are perspective and orthologic, with the center of perspective $P$ and the orthologic centers $Q$ and $Q'$, then the conics $ABCPQ$ and $A'B'C'PQ'$ are equilateral hyperbolas.

Theorem 1.8. (Brianchon - Poncelet [4, pp. 205–220]). The centers of all equilateral hyperbolas passing through the vertices of a triangle $ABC$ lie on the Euler circle of the triangle.

2. Main results

In this section we present two new demonstrations of the theorem of a Belgian mathematician René Goormaghtigh and some consequences deriving from this theorem.

Solution 1. We noted with $A''$ the point of intersection of perpendiculars taken at $B'$ and $C'$ on the $OB, OC$ respectively. Similarly we define the points $B''$ and $C''$ (see Figure 1).
Since $OA = OB = OC$, from the relation
\[
\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = k,
\]
we get $OA' = OB' = OC'$. Because the lines $OA', OB', OC'$ are perpendicular on $B''C''$, $C''A''$, and $A''B''$ respectively, then the point $O$ is the incenter of the triangle $A''B''C''$. Applying theorem 2 in the triangle $A''B''C''$ for the points $A, B, C$, it results that the lines $AA'', BB''$ and $CC''$ are concurrent (at one of Kariya’s points which corresponds to $A''B''C''$ triangle), then triangles $ABC$ and $A''B''C''$ are homological. Thus, according to theorem 3, that the points of intersection of lines $AB$ and $A''B''$, $BC$ and $B''C''$, and $CA$ and $C''A''$ are collinear.

Denote by $X$ the intersection of the lines $BC$ and $B''C''$. Similarly we define the points $Y$ and $Z$.

**Solution 2.** Without restricting the generality suppose that $\angle BCA > \angle ABC$. Let us designate by $R$ the radius of the circle triangle $ABC$, by $A_1$ intersection of the tangent in $A$ to circumcircle of the triangle $ABC$ with the line $CB$, by $T$ and $X'$ the projections of the points $B$ and $X$ on this tangent, by $M$ and $M'$ the projections of points $A'$ and $O$, respectively, with the line $BT$, and by $A'_1$ the intersection of $BC$ and $OM'$ (see Figure 2).

![Figure 2](image_url)

We have: $\angle CAA_1 = \angle ABC$, $\angle ACA_1 = \angle BAC + \angle ABC$ and
\[
\angle AA_1B = 180^\circ - \angle BAC - 2 \cdot \angle ABC
\]
\[
= \angle BCA - \angle ABC, \quad \angle COA'_1 = 2 \cdot \angle ABC - 90^\circ.
\]
Applying the law of sines in the triangle $OCA_1$, we have
\[
\frac{A'_1C}{\sin(2B - 90^\circ)} = \frac{OC}{\sin(C - B)},
\]
so
\[
A'_1C = \frac{-R \cos 2B}{\sin(C - B)}.
\]
Because $XX' = AA' = OA - OA' = R(1 - k)$, then

$\frac{XX'}{\sin(C - B)} = R \frac{1 - k}{\sin(C - B)}$

From $\frac{OA'}{OA} = \frac{A'X}{XA_1} = k$, we get

$\frac{A'X}{XA_1} = \frac{k}{1 - k}$

From relations (1) and (2) we get

$\frac{A_1'X}{XA_1} = \frac{k}{1 - k}$

Since,

$XC = XA_1' + A_1' = \frac{R(k - \cos 2B)}{\sin(C - B)}$

Because $\angle M'OB = 2 \cdot \angle ACB - 90^\circ$, $BM' = BO \cdot \sin(2C - 90^\circ) = -R \cos 2C$, $MM' = OA' = kR$, then $BM = BM' + MM' = R(k - \cos 2C)$. Since

$XB = \frac{BP}{\sin(C - B)} = R \frac{k - \cos 2C}{\sin(C - B)}$

From relations (4) and (5) we get

$\frac{XB}{XC} = \frac{k - \cos 2C}{k - \cos 2B}$

Similarly it is shown that

$YC = \frac{k - \cos 2A}{k - \cos 2C}$

and

$ZA = \frac{k - \cos 2B}{k - \cos 2A}$

We obtain that

$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = 1$

and from the converse of Menelaus’s theorem results that points $X, Y$, and $Z$ are collinear.

**Theorem 2.1.** Let us consider $C_1, C_2, C_3$ and $\mathcal{C}$ the circumcircles of the triangles $AYZ$, $BZX$, $CXY$, and respectively $ABC$. The circles $C_1, C_2, C_3$, and $\mathcal{C}$ pass through a common point.

The proof results from theorem 5.

Let $P$ be the corresponding point of Miquel complete quadrilateral $ABXYCZ$ (see Figure 3).
Theorem 2.2. Centers of circles $C_1, C_2, C_3, \hat{C}$ and point $P$ are on the same circle $\mathcal{K}$.

The proof results from theorems 4 and 5.

Theorem 2.3. Let us consider $C'_1, C'_2,$ and $C'_3$ the circumcircles of the triangles $A''YZ, B''ZX,$ and respectively $C''XY$. The circles $C'_1, C'_2,$ and $C'_3$ pass through a common point.

The proof results from theorem 5.

Let us designate by $O_a, O_b, O_c, O'_a, O'_b, O'_c$ the circumcenters of the triangles $AYZ, BZX, CXY, A''YZ, B''ZX, C''XY,$ respectively, by $Q$ the point of Miquel of $A''C''XZB''Y$ complete quadrilateral (see Figure 4).
Open problems:
1) Point $Q$ is on circle $\mathbb{N}$.
2) Point $O$ is on Aubert’s line of complete quadrilaterals $YABXZC$ and $XZA''C''B''Y$.

Remark 2.1. Goormaghtigh’s theorem is true for $k < 0$, where $\overrightarrow{OA'} = k\overrightarrow{OA}$, $\overrightarrow{OB'} = k\overrightarrow{OB}$, $\overrightarrow{OC'} = k\overrightarrow{OC}$, the demonstration is similar.

Remark 2.2. Points $A'', B''$ and $C''$ are on the perpendicular bisectors of the sides of triangle $ABC$, therefore the triangles $ABC$ and $A''B''C''$ are biogonal, $O$ is a common center of orthology.

Remark 2.3. If $k = 0$ Goormaghtigh’s theorem remains true as a special case of Bobillier’s theorem [10, p.119].
Remark 2.4. For $k = \frac{1}{2}$ we obtain Ayme’s theorem \cite{2}.

Remark 2.5. For $k = 1$ we obtain Lemoine’s theorem and $XYZ$ is Lemoine’s line of the triangle $ABC$ \cite[3. p.155]{3}.

Remark 2.6. Theorem 4 is true for any transversal $XYZ$ which cuts the sides of triangle $ABC$, the demonstration remains the same.

Remark 2.7. Because $O_aO'_a, O_bO'_b, O_cO'_c$ are the perpendicular bisectors of the segments $YZ, ZX, \text{ and } XY$ respectively, then $O_aO'_a \parallel O_bO'_b \parallel O_cO'_c$.

Remark 2.8. The triangles $ABC$ and $A''B''C''$ are perspective, $XYZ$ being the axis of perspective. Let $S$ be the perspective center of triangles $ABC$ and $A''B''C''$.

Theorem 2.4. The lines $OS$ and $XYZ$ are perpendicular.

The proof results by Sondat’s theorem (see Figure 5).

Theorem 2.5. The conics $ABCSO$ and $A'B'C'SO$ are equilateral hyperbolas.

Proof. Because the circumcenter $O$ is the common center of orthology, by Theorem 1.7 we obtain the conclusion. \hfill \Box

Corollary 2.6. The centers of the conics $ABCSO$ and $A'B'C'SO$ lie on the Euler circles of the triangles $ABC$, respectively $A'B'C''$.

The proof results from Theorems 1.8 and 2.5 (see Figure 5).
Two new demonstrations of Goormaghtigh’s theorem

Corollary 2.7. The points $P$ ans $Q$ are the focus of parabolas tangent to the sides of the complet quadrilaterals $ABXYCZ$ and $A''C''XZB''Y$, respectively.

(see [1, p. 109], Figure 5).
3. Dynamic properties

In this section we examine the dependence of considered configuration on homothety coefficient \( k \). Firstly formulate

**Lemma 3.1.** Given two points \( A, B \). The map \( f \) transforms the lines passing through \( A \) to the lines passing through \( B \) and conserve the cross-ratios of the lines. Then the locus of points \( l \cap f(l) \) is a conic passing through \( A \) and \( B \).

Indeed if \( X, Y, Z \) are three points of the thought locus, then lines \( l \) and \( f(l) \) intersect the conic \( ABXYZ \) in the same point.

Lemma 3.1 has also a dual formulating: if \( f \) is a projective map between lines \( a \) and \( b \) then the envelop of lines \( Af(A) \) is a conic touching \( a \) and \( b \).

Using Lemma 3.1 we obtain that the envelop of lines \( XYZ \) from Theorem 1.1 is a parabola touching the sidelines of \( ABC \), and the locus of perspectivity centers from Theorem 1.2 is the Feuerbach hyperbola.

**Theorem 3.1.** Point \( P \) from Theorem 2.1 is fixed.

**Proof.** Immediately follows from Lemma 3.1 and Corollary 2.7.

**Theorem 3.2.** The locus of points \( Q \) is a line passing through \( O \).

**Proof.** Using polar transformation with center \( O \) we obtain from Theorem 2.5 that two parabolas from Corollary 2.7 are homothetic. Thus all points \( Q \) are the foci of homothetic parabolas.

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**References**


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