INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 1 (2012), No. 1, 5 - 9

A NEW PROOF OF NEUBERG'S THEOREM AND ONE APPLICATION

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Abstract. In this paper, we will give a new proof of Neuberg's Theorem. One application of this result is found in Theorem 2.2.

1. Introduction

In this section, we'll recall some known results.

Theorem 1.1. *Let ABC be a triangle, $\mathcal{C}(O, R)$ his circumscribed circle and H the orthocentre of this triangle. If M is the midpoint of the side BC, then*

(1) $$AH = 2OM$$

and

(2) $$OM \perp BC.$$

For additional comments and proofs, see [1]-[4].

In this paper, we will consider a convex quadrilateral $ABCD$ and note with H_a, H_b, H_c, H_d the orthocentres of triangles BCD, CDA, DAB, respectively ABC and with $T[ABCD]$ the area of quadrilateral $ABCD$.

Theorem 1.2. *If $ABCD$ is a cyclic quadrilateral, then*

(3) $$AB \equiv H_aH_b,\ AB \| H_aH_b,$$

(4) $$AH_b \perp CD,\ BH_a \perp CD,$$

(5) $$AH_b \equiv BH_a$$

and analogues.

For comments and proofs see [3].

Keywords and phrases: Convex quadrilateral, area
(2010)Mathematics Subject Classification: 51M04, 51M25, 51M30

Corollary 1.3. *If $ABCD$ is a cyclic quadrilateral, then*

$$T[ABCD] = T[H_aH_bH_cH_d]. \tag{6}$$

Proof. It results from Theorem 1.2. □

Corollary 1.4. *Let $ABCD$ be a convex quadrilateral. If $AH_b \equiv BH_a$, $BH_c \equiv CH_b$, $CH_d \equiv DH_c$ or $DH_a \equiv AH_d$, then $ABCD$ is a cyclic quadrilateral.*

Proof. Let's consider the relation $AH_b \equiv BH_a$.Let $\mathcal{C}(O_1)$ and $\mathcal{C}(O_2)$ be the circumscribed circles of the triangle ADC, respectively BCD and M the midpoint of DC. Taking Theorem 1.1 into account, we have that

$$AH_b = 2O_1M,\ BH_a = 2O_2M,$$

and

$$O_1M \perp DC, O_2M \perp DC,$$

from where, O_1 coincide to O_2. So, the quadrilateral $ABCD$ is cyclic. □

2. Main results

In this section, the main results are proved by using the analytic geometry. The result from Theorem 2.1 is known in literature as Neuberg's Theorem (see [4] or [6]) and Theorem 2.2 is an application of this.

Theorem 2.1. *If M is a point in the same plan with the triangle ABC, $M \notin AB \cup BC \cup CA$ and H_a, H_b, H_c are the orthocentres of triangles MBC, MCA, respectively MAB, then*

$$T[ABC] = T[H_aH_bH_c]. \tag{7}$$

Proof. We consider $a, b, c, \lambda, \mu \in \mathbb{R}$, $a > 0$, $b < c$ and the points $A(O, a)$, $B(b, 0)$, $C(c, 0)$ and $M(\lambda, \mu)$ (Figure 1). From $M \notin BC$, it results that $\mu \neq 0$. The equation of the line AB is

$$ax + by - ab = 0,$$

and then, from $M \notin AB$ it results that

$$a\lambda + b\mu - ab \neq 0.$$

Similarly, from $M \notin AC$ it results that

$$a\lambda + c\mu - ac \neq 0.$$

Taking the remarks above, from $M \notin AB \cup BC \cup CA$ we obtain that

$$\mu(a\lambda + b\mu - ab)(a\lambda + c\mu - ac) \neq 0. \tag{8}$$

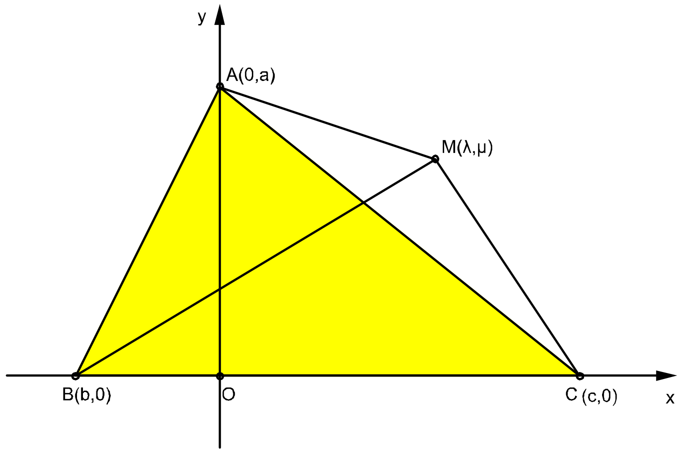

Figure 1

Writing the corresponding heights equations, after the calculations, we obtain

$$H_a\left(\lambda, \frac{-\lambda^2+(b+c)\lambda-bc}{\mu}\right), \tag{9}$$

$$H_b\left(\frac{\mu(c\lambda-a\mu+a^2)}{a\lambda+c\mu-ac}, \frac{\lambda(a\mu-b\lambda+b^2)}{a\lambda+b\mu-ab}\right) \tag{10}$$

and

$$H_c\left(\frac{\mu(b\lambda-a\mu+a^2)}{a\lambda+b\mu-ab}, \frac{\lambda(a\mu-b\lambda+b^2)}{a\lambda+b\mu-ab}\right). \tag{11}$$

If $\lambda = 0$, then from $M \notin AB \cup CA$ it results that $bc \neq 0$. In this case $H_a\left(0, -\frac{bc}{\mu}\right)$, $H_b\left(-\frac{a\mu}{c}, 0\right)$, $H_c\left(-\frac{a\mu}{b}, 0\right)$ and then $T[H_aH_bH_c] = |\Delta|$, where

$$\Delta = \frac{1}{2}\begin{vmatrix} x_{H_a} & y_{H_a} & 1 \\ x_{H_b} & y_{H_b} & 1 \\ x_{H_c} & y_{H_c} & 1 \end{vmatrix},$$

$H_a(x_{H_a}, y_{H_a})$ and analogues. We have that

$$\Delta = \frac{1}{2}a(b-c),$$

so

$$T[H_aH_bH_c] = T[ABC]$$

and then, in this case the theorem was proved. If $\lambda \neq 0$, we note

$$E = (a\lambda + c\mu - ac)(a\lambda + b\mu - ab)$$

and then

$$E = a^2\lambda^2 + a\lambda(\mu - a)(b+c) + bc(\mu - a)^2. \tag{12}$$

Taking (8) and (12) into account, we have that $\mu E \neq 0$.
Then

$$\Delta = \frac{1}{2\mu E}\begin{vmatrix} \lambda\mu & -\lambda^2+(b+c)\lambda-bc & \mu \\ \mu(c\lambda-a\mu+a^2) & \lambda(a\mu-c\lambda+c^2) & a\lambda+c\mu-ac \\ \mu(b\lambda-a\mu+a^2) & \lambda(a\mu-b\lambda+b^2) & a\lambda+b\mu-ab \end{vmatrix}$$

and multiplying the first line with λ and dividing the second column with λ, we obtain

$$\Delta = \frac{1}{2E}\begin{vmatrix} \lambda^2 & -\lambda^2+(b+c)\lambda-bc & \lambda\mu \\ c\lambda-a\mu+a^2 & a\mu-c\lambda+c^2 & a\lambda+c\mu-ac \\ b\lambda-a\mu+a^2 & a\mu-b\lambda+b^2 & a\lambda+b\mu-ab \end{vmatrix}.$$

Adding the first column to the second column, we obtain

$$\Delta = \frac{1}{2E}\begin{vmatrix} \lambda^2 & (b+c)\lambda-bc & \lambda\mu \\ c\lambda-a\mu+a^2 & a^2+c^2 & a\lambda+c\mu-ac \\ b\lambda-a\mu+a^2 & a^2+b^2 & a\lambda+b\mu-ab \end{vmatrix}.$$

Deducting the third line from the second line, we get

$$\Delta = \frac{c-b}{2E}\begin{vmatrix} \lambda^2 & (b+c)\lambda-bc & \lambda\mu \\ \lambda & c+b & \mu-a \\ b\lambda-a\mu+a^2 & a^2+b^2 & a\lambda+b\mu-ab \end{vmatrix}.$$

We multiply the second line with $-\lambda$ and adding it to the first line, and we multiply the second line with $-b$ and adding it to the third line, then we obtain

$$\Delta = \frac{c-b}{2E}\begin{vmatrix} 0 & -bc & a\lambda \\ \lambda & b+c & \mu-a \\ -a(\mu-a) & a^2-bc & a\lambda \end{vmatrix}.$$

Deducting the first line from the third line and taking that

$$T[ABC] = \frac{a(c-b)}{2},$$

we have

$$\begin{aligned} \Delta &= \frac{c-b}{2E}\begin{vmatrix} 0 & -bc & a\lambda \\ \lambda & b+c & \mu-a \\ -a(\mu-a) & a^2 & 0 \end{vmatrix} \\ &= \frac{1}{E}T[ABC]\begin{vmatrix} 0 & -bc & a\lambda \\ \lambda & b+c & \mu-a \\ -(\mu-a) & a & 0 \end{vmatrix} \\ &= \frac{1}{E}T[ABC]E = T[ABC]. \end{aligned}$$

Now, the relation (7) is proved. □

Theorem 2.2. *If $ABCD$ is a convex quadrilateral, then*

$$T[ABCD] = T[H_aH_bH_cH_d]. \tag{13}$$

Proof. Any three from the points H_a, H_b, H_c, H_d are not collinear. Even if $H_aH_bH_cH_d$ is a concave quadrilateral, it has at least one diagonal situated on the surface of the quadrilateral $ABCD$, for example $[H_bH_d]$ (Figure 2).Applying Theorem 2.1 for the triangle ABD and for the point C, for the

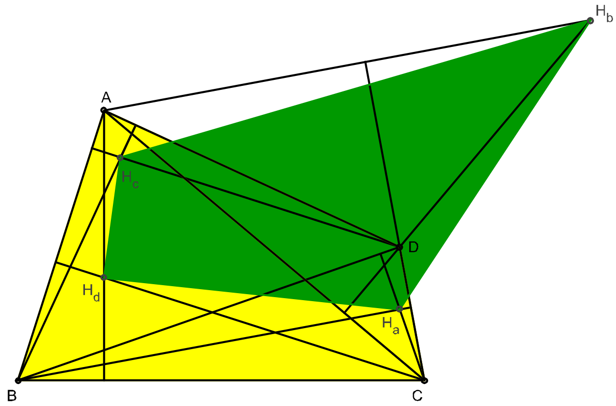

Figure 2

triangle CBD and for the point A respectively, we obtain that

$$T[ABD] = T[H_aH_bH_d],$$

respectively

$$T[CBD] = T[H_cH_bH_d].$$

From the equalities above, (13) follows. □

Remark 2.3. Taking Theorem 2.2 into account, it results that (13) takes place also in quadrilateral that are not cyclic quadrilaterals.

References

[1] Barbu, C., *Fundamental Theorems of Triangle Geometry* (Romanian), Ed. Unique, Bacău, 2008.

[2] F. G.-M., *Exercices de géométrie*, Huitième édition, Paris, 1931.

[3] Golovina, L. I. and Iaglom, I. M., *Induction in Geometry* (Romanian), Ed. Tehnică, Bucharest, 1964.

[4] Miculiţa, M. and Brânzei, D., *Analogies Triangle-Tetrahedron* (Romanian), Ed. Paralela 45, 2000.

[5] Mihalescu, C., *The Geometry of Remarkable Elements* (Romanian), Ed. Tehnică, Bucharest, 1957.

[6] Neuberg, J., *Mathesis* (1881), 168.

[7] Pop, O. T., Minculete, N. and Bencze, M., *An Introduction to Quadrilateral Geometry* (to appear).

Received: December 16, 2011.

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