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On Generalizations of Bundle Theorem and Miquel's Six Circles Theorem on the Plane

KIEN TRUNG NGUYEN, SOMNUEK WORAWISET,
TRAN THU LE

Abstract. Bundle theorem and Miquel's six circles theorem are two well-known theorems that concern the property of a system of points on the plane, where some of quadruples of points are concyclic. In this paper we propose generalizations for these two theorems. We call a generalization of bundle theorem the first six conics theorem and a generalization of Miquel's six circles theorem the second six conics theorem, as they are both related to six conics in their configuration. Based on these two generalizations, interesting results are also discussed.

1. Introduction

We investigate the property concerning 8 points and six circles on the plane in the setting of *bundle theorem*; see Hartmann [2]. In general, bundle theorem regards a property of a Möbius plane that is fulfilled by ovoidal Möbius planes only; see Kahn [5]. In 1996, Santos [4] considered this theorem in normed space and proved that if the bundle theorem holds in a strictly convex and smooth normed plane, then the corresponding plane is Euclidean.

Another well-known theorem in Euclidean geometry, which is closely related to bundle theorem, is the so-called *Miquel's six circles theorem* (MSCT); see Pedoe [7].

In 1960 Asplund and Grünbaum [1] investigated MSCT in normed space. They stated that, in a strictly convex and smooth normed plane MSCT holds if all of the six circles have the same radius. Margarita Spirova [8] further proved that MSCT is also correct on arbitrary normed plane, i.e., the normed plane not requiring the smoothness and strict-convexcity. In addition, if MSCT holds in a strictly convex and smooth normed plane, then this plane is Euclidean. Yaglom [6] showed in 1979 that, MSCT is also correct if we consider the similar setting of the theorem on the Minkowskian plane and the Galelian plane.

We investigate in this paper the generalizations of the two theorems, say bundle theorem and MSCT, based on a projective approach. We know

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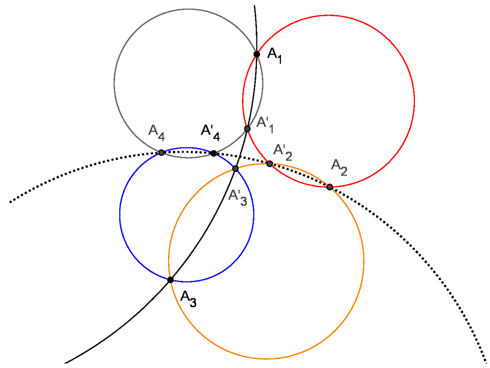

FIGURE 1. Bundle theorem

that in the Euclidean model of projective plane, every circle always passes through two cyclic points; see Gallier [3]. Therefore, if we choose an appropriate projective basis so that the two points contained in a conic S become two cyclic points, then S is a circle. We apply this property to generalize some results in the planar geometry. Precisely, by adding two cyclic points to the setting of bundle theorem and MSCT, we obtain some results that are extensions of these two theorems.

The paper is organized as follows. Section 2 considers the first six conics theorem, which is a generalization of bundle theorem. Moreover, we study in Section 3 a generalization of Miquel's six circles theorem, say the second six conics theorem. Interesting results derived from the first and the second six conics theorems are also discussed.

2. The First Six Conics Theorem - A Generalization of Bundle Theorem

Let us first review the so-called bundle theorem as follows.

Theorem 2.1. *(Bundle theorem) If eight points A_i, A'_i for $i = 1, \dots, 4$ such that five of six quadruples $\{A_i, A'_i, A_j, A'_j\}$ are concyclic, then the sixth quadruple is also concyclic.*

As mentioned above, every circle always passes through two cyclic points. Therefore, the bundle theorem can be generalized as the following one.

Theorem 2.2. *(First six conics theorem) Given ten points A_i, A'_i, M, N for $i = 1, \dots, 4$ such that five of hexatruples $A_i, A'_i, A_j, A'_j, M, N$ with $i, j \in \{1, \dots, 4\}$ and $i \neq j$ are inscribed in a conic, then the sixth one is inscribed another conic.*

To prove this theorem, let us first restate the so-called three conics theorem:

Theorem 2.3. *(Three conics theorem) If three conics pass through two given points, then the line joining the others two intersections of each pair of conics are concurrent.*

Furthermore, we state the inverse version as in what follows:

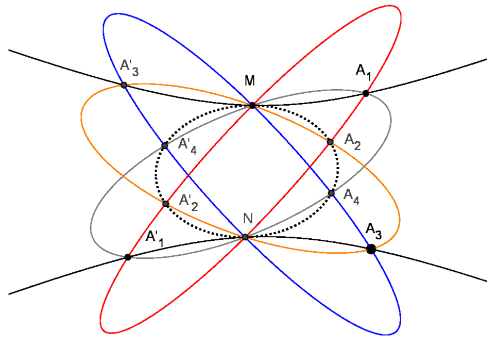

FIGURE 2. The first six conics theorem

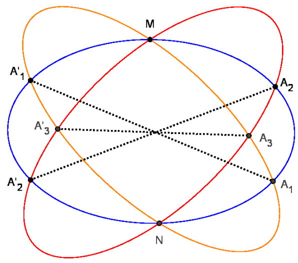

FIGURE 3. Three conics theorem. S_1-red; S_2-orange

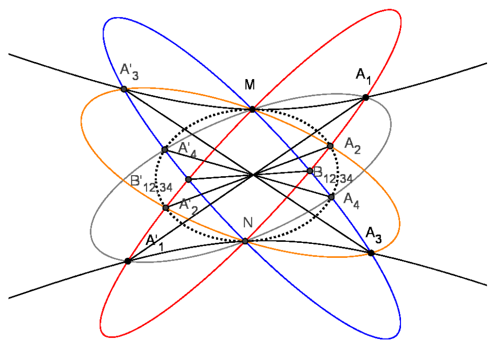

FIGURE 4. S_{12}-red; S_{23}-orange; S_{34}-blue; S_{41}-grey; S_{13}-black; S_{24}-dash

Theorem 2.4. *(Inverse of three conics theorem) Two conics S_1, S_2 meet at four points M, N, A_3, A'_3. If $A_1A'_1$ and $A_2A'_2$ are chords of S_1 and S_2 respectively which meet on $A_3A'_3$, then the six points $M, N, A_1, A'_1, A_2, A'_2$ lie on a conic.*

For illustration of the three conics and inverse three conics theorems, one can refer to Figure 3

We now apply the three conics and inverse three conics theorem to prove Theorem 2

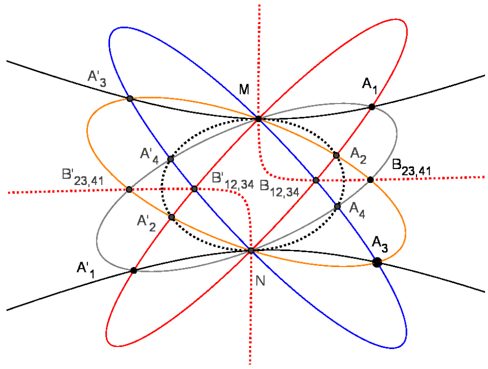

FIGURE 5. S_{12}-red; S_{23}-orange; S_{34}-blue; S_{41}-grey; S_{13}-black; S_{24}-dashed black; S'-dashed red.

Proof. Without loss of generality, suppose that there are five conics S_{ij} which passes through A_i, A'_i, A_j, A'_j, M, N for $(i,j) \in \{(1,2);(2,3);(3,4);(4,1);(1,3)\}$. We prove that $A_2, A'_2, A_4, A'_4, M, N$ are inscribed in a conic, say S_{24}. Denote by $B_{12,34}, B'_{12,34}$ the other intersections, except M, N, of S_{12}, S_{34}. Applying the three conics theorem for S_{12}, S_{23}, S_{41} , the four lines $A_2A'_2, A_3A'_3, B_{12,34}, B'_{12,34}$ are concurrent. By the same argument for S_{12}, S_{41}, S_{34} and S_{12}, S_{13}, S_{34} , and we also obtain $A_1A'_1$, $A_4A'_4$, $B_{12,34}, B'_{12,34}$ and $A_1A'_1, A_3A'_3, B_{12,34}B'_{12,34}$ concurrent, respectively. Therefore, $A_1A'_1, A_2A'_2, A_3A'_3, A_4A'_4$ and $B_{12,34}B'_{12,34}$ are concurrent. By the inverse three conics theorem and the concurrency of $A_2A'_2$, $A_4A'_4$, $B_{12,34}B'_{12,34}$, the points $A_2, A'_2, A_4, A'_4, M, N$ are inscribed in a conic.

By exchanging the role of S_{23} and S_{34} in the proof of Theorem 2, we obtain another version as follows.

Proposition 2.1. *Denote $B_{12,34}, B'_{12,34}$ are the others intersections of S_{12} and S_{34}. Denote $B_{23,41}, B'_{23,41}$ are the others intersections of S_{23} and S_{41}. Then $B_{12,34}, B'_{12,34}, B_{23,41}, B'_{23,41}, M, N$ are inscribed in a conic S'. (See Fig. 5)*

In Proposition 2.1 let M and N be two cyclic points, then all conics in the proposition are circles. We get another circle in the structure of bundle theorem.

Proposition 2.2. *If eight points A_i, A'_i for $i = 1, \ldots, 4$ such that five of six quadruples A_i, A'_i, A_j, A'_j are inscribed in a circle S_{ij} with $i, j \in \{1, \ldots, 4\}$ and $i \neq j$. Let the intersections of S_{12} and S_{34} be $B_{12,34}, B'_{12,34}$. Also, denote the intersections of S_{23} and S_{41} by $B_{23,41}, B'_{23,41}$. Then $B_{12,34}, B'_{12,34}, B_{23,41}, B'_{23,41}$ inscribed in a circle S'.*

3. THE SECOND SIX CONICS THEOREM - A GENERALIZATION OF MIQUEL'S SIX CIRCLES THEOREM

For simplicity an index $i \geq 4$ coincides with i mod 4 $(1 \leq i \leq 4)$. We first restate Miquel's six circles theorem (MSCT) as follows

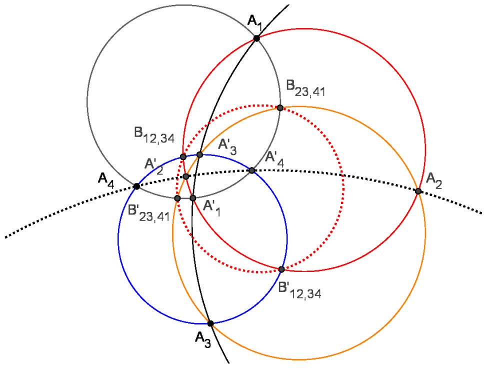

FIGURE 6. Bundle theorem was completed with a seventh circle S'. S_{12}-red; S_{23}-orange; S_{34}-blue; S_{41}-grey; S_{13}-black; S_{24}- dashed black; S'- red dash

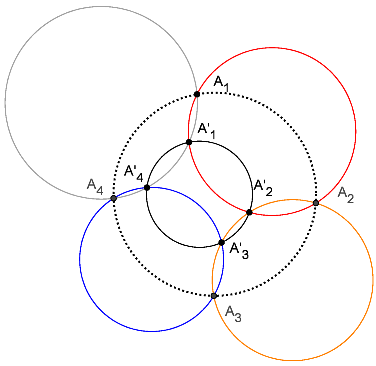

FIGURE 7. Miquel's six circles theorem

Theorem 3.1. *(Miquel's six circles theorem) If eight points A_i, A'_i for $i = 1, \ldots, 4$ such that four quadruples $A_i, A'_i, A_{i+1}, A'_{i+1}$ and A_1, A_2, A_3, A_4 are concyclic, then A'_1, A'_2, A'_3, A'_4 are also concyclic.*

MSCT can be illustrated by Fig. 7 MSCT can be generalized by the following result.

Theorem 3.2. *(Second Six Conics Theorem) If ten points A_i, A'_i, M, N for $= 1, \ldots, 4$ such that four hexatruples $A_i, A'_i, A_{i+1}, A'_{i+1}, M, N$ are inscribed in conic $S_{i(i+1)}$ for $i = \overline{1,4}$ and A_1, A_2, A_3, A_4, M, N are inscribed in conic S, then $A'_1, A'_2, A'_3, A'_4, M, N$ are inscribed in another conic.*

Proof. For $i \neq j$, Denote the equation of A_iA_j and S_{ij} by $L_{ij} = 0$ and $S_{ij} = 0$, respectively. Also, the equation of conic S is $S = 0$. By choosing a suitable projective basis, we obtain $A_1 = (0, 0, 1), A_2 = (0, 1, 0), A_3 = (1, 0, 0), A_4 = (1, 1, 1), M = (m, n, p)$. These five points determine exactly the conic S. Assume that the equation of S is

$$S = ax^2 + by^2 + cz^2 + dxy + eyz + fzx = 0.$$

As S consists of five points A_1, A_2, A_3, A_4, M, the equation of S becomes

$$S = dxy + eyz + fzx = 0.$$

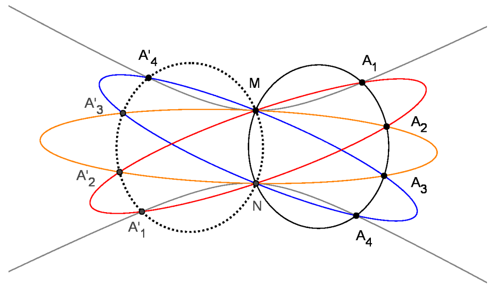

FIGURE 8. The second six conics theorem. S-black; S_{12}-red; S_{23}-orange; S_{34}-blue; S_{41}-grey; S'-dashed black

Here, d, e, f satisfy the following equations

$$\begin{cases} d + e + f = 0. \\ dmn + enp + fpm = 0. \end{cases} \tag{1}$$

Similarly, we obtain the equations of A_iA_j for $i \neq j$ as follows.

$$\begin{aligned} L_{12} &= x = 0. \\ L_{23} &= z = 0. \\ L_{34} &= y - z = 0. \\ L_{41} &= -x + y = 0. \end{aligned}$$

As S and S_{ij} intersect at four points A_i, A_j, M, N and MN, A_iA_j (with equations $L = 0$ and $L_{ij} = 0$) also pass through these points, we get S_{ij} as a linear combination of S and LL_{ij}

$$S_{ij} = a_{ij}S + b_{ij}LL_{ij}.$$

Because S_{ij} differ from LL_{ij}, hence $a_{ij} \neq 0$. By assigning $S_{ij} := \dfrac{S_{ij}}{a_{ij}}$ and $L_{ij} := c_{ij}L_{ij}$, where $c_{ij} := \dfrac{b_{ij}}{a_{ij}}$, We obtain

$$S_{ij} = S + LL_{ij}$$

and

$$\begin{aligned} L_{12} &= c_{12}x \\ L_{23} &= c_{23}z \\ L_{34} &= c_{34}(y - z) \\ L_{41} &= c_{41}(-x + y) \end{aligned}$$

Now, we prove that $S = L_{ij}L_{kl} + L_{jk}L_{li}$. Indeed, assume it is correct, we have

$$\begin{aligned}
&S = L_{12}L_{34} + L_{23}L_{41}\\
\Leftrightarrow\ &dxy + eyz + fzx = c_{12}xc_{34}(y - z) + c_{23}zc_{41}(-x + y)\\
\Leftrightarrow\ &dxy + eyz + fzx = c_{12}c_{34}xy + c_{23}c_{41}yz - (c_{12}c_{34} + c_{23}c_{41})xz
\end{aligned}$$

We trivially choose d, e, f such that $d = c_{12}c_{34}$, $e = c_{23}c_{41}$, $f = -c_{23}c_{41} - c_{23}c_{41}$. Obviously, the coefficents d, e, f such that (1) holds or $S = L_{ij}L_{kl} + L_{jk}L_{li}$ We now focus on the theorem. Remind that S_{li}, S_{ij} pass through A_i, A'_i, M, N and L passes through M, N. By $S_{li} - S_{ij} = L(L_{li} - L_{ij})$, we observe that $L_i := L_{li} - L_{ij}$ contain A_i and A'_i. Consider

$$\begin{aligned}
&S_{ij} + L_iL_j\\
=&S + LL_{ij} + (L_{li} - L_{ij})(L_{ij} - L_{jk})\\
=&L_{ij}L_{kl} + L_{jk}L_{li} + LL_{ij} + L_{li}L_{ij} - L_{li}L_{jk} - L_{ij}^2 + L_{ij}Ljk\\
=&L_{ij}(L - L_{ij} + L_{jk} + L_{kl} + L_{li})\\
=&L_{ij}(T - 2L_{ij}),
\end{aligned}$$

where $T = L + L_{ij} +_{jk} +L_{kl} + L_{li}$. Assigning: $L'_{ij} := T - 2L_{ij}$, we have $S_{ij} + L_iL_j = L_{ij}L'_{ij}$. As S_{ij} and L_iL_j both pass through A_i, A_j, A'_i, A'_j, and L_{ij} passes through A_i, A_j, then L'_{ij} contains A'_i, A'_j.
Denote by S', $S' = L'_{ij}L'_{kl} + L'_{jk}L'_{li}$. We prove S' is a conic passing through $A'_1, A'_2, A'_3, A'_4, M, N$. Indeed, we get $A'_i \in S'$ as A'_i is in L'_{li} and L'_{ij} for $i = \overline{1, 4}$. We just prove that S' contains M, N. We have

$$\begin{aligned}
S' &= L'_{ij}L'_{kl} + L'_{jk}L'_{li}\\
&= (T - 2L_{ij})(T - 2L_{kl}) + (T - 2L_{jk})(T - 2L_{li})\\
&= 2T^2 - 2T(L_{ij} + L_{jk} + L_{kl} + L_{li}) + 4(L_{ij}L_{kl} + L_{jk}L_{li})\\
&= 2T^2 - 2T(T - L) + 4S\\
&= 2LT + 4S.
\end{aligned}$$

It shows clearly that S' passes through M, N as L and S pass through M, N. The theorem has been proved.

Next, we get some corollaries concerning this theorem as below

Corrolary 3.1. *Given two conics S and S' intersecting at two points, say M, N. An arbitrary line passes through M and cuts S, S' at A_2, A'_3, respectively. Another line passes through N cuts S, S' at A_3, A'_2, respectively. Suppose that MA_3 and NA_2 cut S' at A'_4 and A'_1, respectively. Moreover, MA'_3, MA'_2 cut S again at A_4, A_1, respectively. Then $A_1, A'_1, A_4, A'_4, M, N$ are inscribed in a conic.*

Proof. This corollary is derived directly from Theorem 3.2 if the conics S_1, S_2, S_3 are degenerated into pairs of lines.

Corrolary 3.2. *If three conics S_{12}, S_{34}, S passes through M, N and S_{12}, S_{34} tangent to S at A_{12}, A_{34}, respectively. The conic S_{23} meets S_{12}, S_{34} again at A'_{12}, A'_{34}. Then there exists a conic S' passing through M, N and tangent to S_{12}, S_{34} at A'_{12}, A'_{34}.*

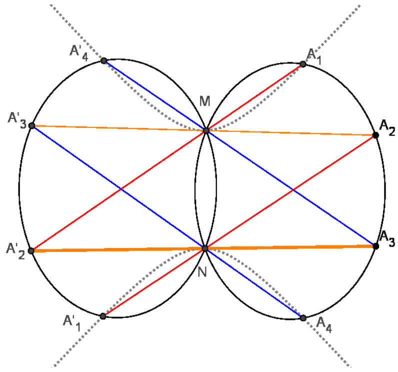

FIGURE 9. S and S'-black; S_{12}-red; S_{23}-orange; S_{34}-blue; S_{41}-dashed grey

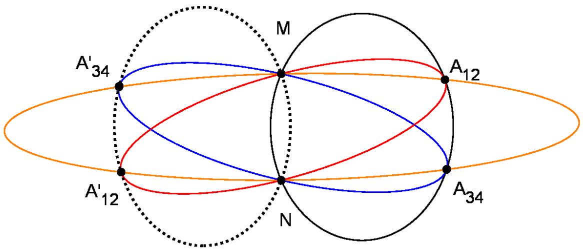

FIGURE 10. S-black; S_{12}-red; $S_{23} \equiv S_{41}$-orange; S_{34}-blue; S'-dashed black

Proof. In Theorem 3.2, let $A_1 \equiv A_2 \equiv A_{12}$, $A_3 \equiv A_4 \equiv A_{34}$, $A'_1 \equiv A'_2 \equiv A'_{12}$, $S_{23} \equiv S_{41}$. The result follows.

Now we define two additional points of a quaterlateral $ABCD$ on the projective plane as the intersections of AB and CD, BC and DA. We obtain the following corollary.

Corrolary 3.3. *Given six points A_i, M, N for $i = 1, \dots, 4$ on a conic S. Additional points of A_1MA_3N, A_2MA_4N and two points M, N lie on a conic.*

Proof. Let A'_1, A'_3 be the intersections of A_2M and A_4N, MA_4 and NA_2, respectively. Similarly, let the intersections of A_3M and A_1N, A_1M and A_3N be A'_2, A'_4. Moreover, the conics $S_{12}, S_{23}, S_{34}, S_{41}$ in Theorem 3.2 are degenarated into couples of lines. We obtain the corrollary.

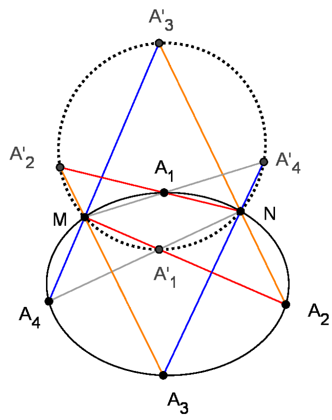

FIGURE 11. S-black; S_{12}-red; S_{23}-orange; S_{34}-blue; S_{41}-grey; S'-dashed black

4. Conclusions

We generalized bundle theorem and MSCT by a projective approach. Based on three conics theorem, we derive the so-called first six conics theorem which is a generalizations of bundle theorem. Moreover, by choosing an appropriate projective basis, we obtain the second six conics theorem - a generalization of MSCT. Also, we discuss corrollaries related to these two generalizations. It is promising to apply the technique of adding two cyclic points to existing models in order to generalize some further results, which are related to circles in Euclidean plane.

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"CAN THO" UNIVERSITY
DEPARTMENT OF MATHEMATICS, TEACHER COLLEGE
NO.1, STREET 3/2, CAN THO, VIETNAM
E-mail address: trungkien@ctu.edu.vn
E-mail address: letranthu23041996@gmail.com

"KHON KEAN" UNIVERSITY
DEPARTMENT OF MATHEMATICS, SCIENCE COLLEGE
KHON KEAN, THAILAND
E-mail address: wsomnu@kku.ac.th