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NEW PARTNER CURVES IN THE EUCLIDEAN 3-SPACE E^3

ONUR KAYA and MEHMET ÖNDER

Abstract. In this paper, first, an alternative frame and alternative curvatures of a space curve are summarized. Later, a new type of special curve pairs is defined as CN^*-partner curves and characterizations for these curves are introduced. The distance function between the corresponding points of reference curve and its partner curve and the angle function between the vector fields of alternative frame of the curves are obtained by means of alternative curvatures. Also, some relations between CN^*-partner curves and some special curves such as helices and slant helices are given. Finally, some examples are introduced.

1. Introduction

The space curves for which there exist some relations between Frenet frames and curvatures of the curves are the most fascinating subject of curve theory. The most famous types of such curves are Bertrand curves. These curves have been obtained after a question asked by French mathematician Saint-Venant in 1845 [13]. The question is that whether upon the ruled surface generated by the principal normals of a curve in the three-dimensional Euclidean space E^3, is there another curve such that the principal normals of original curve are also its principal normals? The answer of this question was introduced by Bertrand. He obtained that such a curve exists if and only if a linear relation with constant coefficients shall exists between the curvature and torsion of original curve [2]. Later, such curve pairs have been called Bertrand partner curves or Bertrand curves and these special curves have been studied by many mathematicians and different characterizations and applications of these curves have been introduced [1, 3, 6, 7, 10, 11, 12, 17].

Recently, Liu and Wang have introduced a new definition of curve pairs by originating the notion of Bertrand curve. They have called these new kind of curve pairs as Mannheim partner curves given by the property that the principal normal vectors of original curve coincide with the binormal vectors of second curve [16]. After, Mannheim curves have been studied by

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some mathematicians and new properties of these curves have been obtained [4, 9, 18].

The goal of this paper is to define a new kind of associated curve pairs and give characterizations for these curves. For this purpose, we use an alternative frame on space curves and define a special curve pair by using this frame. This new curve pair is called CN^*-partner curves. First, the definition and main characterizations related to distance function and angle function of CN^*-partner curves are introduced. Later, some relationships between CN^*-partner curves and some special curves such as helix and slant helix are obtained. In the last section, some numerical examples are given.

2. Preliminaries

Let $\alpha = \alpha(s)$ be a regular unit speed curve in the Euclidean 3-space E^3 and $\{T, N, B\}$ be the Frenet frame of $\alpha(s)$, where T, N, B are unit tangent vector field, principal normal vector field and binormal vector field, respectively. Then the Frenet formulae of the curve is given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where κ, τ are called the curvature and the torsion of the curve, respectively. From (1), the unit Darboux vector W of $\alpha(s)$ given by the equation

$$W = \frac{1}{\sqrt{\kappa^2 + \tau^2}} (\tau T + \kappa B) \tag{2}$$

is the angular velocity vector of the curve α [8]. It is obvious from (2) that the Darboux vector is perpendicular to the principal normal vector field N. Then, defining a unit vector field C by the cross product $C = W \times N$ makes it possible to build another orthonormal moving frame along the curve $\alpha(s)$. This frame is represented by $\{N, C, W\}$ and is an alternative frame to curve rather than the Frenet frame $\{T, N, B\}$.

The derivative formulae of the alternative frame is given by

$$\begin{bmatrix} N' \\ C' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & \beta & 0 \\ -\beta & 0 & \gamma \\ 0 & -\gamma & 0 \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix}, \tag{3}$$

where $\beta = \sqrt{\kappa^2 + \tau^2}$ and $\gamma = \frac{\kappa^2}{\kappa^2+\tau^2} \left(\frac{\tau}{\kappa}\right)'$ [15]. Since the principal normal vector N is common in both frames, it is possible to form a relationship between the Frenet frame and alternative frame such as

$$\begin{cases} C = -\bar{\kappa} T + \bar{\tau} B, \\ W = \bar{\tau} T + \bar{\kappa} B, \end{cases} \tag{4}$$

or

$$\begin{cases} T = -\bar{\kappa} C + \bar{\tau} W, \\ B = \bar{\tau} C + \bar{\kappa} W, \end{cases} \tag{5}$$

where $\bar{\kappa} = \kappa/\beta$ and $\bar{\tau} = \tau/\beta$.

A regular curve α is called a helix if the tangent lines of the curve make a constant angle with a fixed direction and a helix is characterized by the property that $\frac{\tau}{\kappa}$ is constant [14]. If the principal normal lines of the curve make a constant angle with a fixed direction, then the curve is called a slant helix and characterized by the equality

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' = constant.$$

[8]. Then, the characterization of a slant helix according to alternative frame is given as follows:

Remark 2.1. *A regular curve α with alternative curvatures β, γ is a slant helix if and only if $\frac{\gamma}{\beta}(s) = \text{constant}$.*

3. CN^*-Partner curves in E^3

The curve pairs for which there exists a relation between the Frenet vectors of the curves are an important study for the characterizations of space curves. Some examples of such curve pairs are Bertrand partner curves, Mannheim partner curves and invoule-evolute curves. Two curves having a common principal normal vector are called Bertrand curves. The classical characterization for Bertrand curves is that a regular curve α in E^3 is a Bertrand curve if and only if $a\kappa(s) + b\tau(s) = 1$ holds where κ and τ are curvature and torsion of curve and a, b are constant real numbers [11]. Moreover, Bertrand curves are characterized with constant distance between the corresponding points of curves and with constant angle between tangent vector fields of curves [14].

Another type of curve pairs is Mannheim partner curves. Two curves α and ϑ in E^3 are called Mannheim partner curves or Mannheim curve pair if the principal normal vector fields of α coincide with the binormal vector fields of ϑ at the corresponding points of curves. Mannheim partner curves have also characterizations similar to Bertrand partner curves. But they have some difference. For instance, although the angle between unit tangent vector fields of Bertrand curves is constant, it is not constant for Mannheim partner curves [9, 16].

In this section, a new type of curve pairs is defined by considering alternative frame and characterizations for these curve pairs are introduced by means of alternative curvatures. Two functions are considered for these characterizations such as distance function and angle function. First, we give the following definition.

Definition 3.1. *Let $\alpha = \alpha(s)$ and $\alpha^* = \alpha^*(s^*)$ be two regular space curves in the Euclidean 3-space E^3 with Frenet frames $\{T, N, B\}$, $\{T^*, N^*, B^*\}$, curvatures κ, κ^*, torsions τ, τ^*, respectively, and let the alternative moving frames and alternative curvatures of curves be $\{N, C, W\}$, β, γ and $\{N^*, C^*, W^*\}$, β^*, γ^*, respectively. The curves α and α^* are called CN^*-partner curves or $\{\alpha, \alpha^*\}$ is called a CN^*-curve pair if the vector fields C and N^* coincide, i.e., $C = N^*$ holds at the corresponding points of the curves.*

By considering Definition 3.1, one can easily write the parametric representation of α^* as follows

$$\alpha^*(s) = \alpha(s) + R(s)C(s),$$

where $R = R(s)$ is the distance function between corresponding points of the curves α and α^*. Since the vector fields C and N^* are the same, we are able to give the relationship between the alternative frames of α and α^* such as

$$\left\{ \begin{array}{l} C^* = \sin\theta N + \cos\theta W, \\ W^* = \cos\theta N - \sin\theta W, \end{array} \right. \tag{6}$$

or

$$\left\{ \begin{array}{l} N = \sin\theta C^* + \cos\theta W^*, \\ W = \cos\theta C^* - \sin\theta W^*. \end{array} \right. \tag{7}$$

where $\theta = \theta(s)$ is the angle function between vector fields N and W^*.

Now, we give some characterizations for CN^*-partner curves. Whenever we talk about the curves α and α^*, we will assume that the curves have frames and curvatures as given in Definition 3.1.

Theorem 3.1. *Let $\{\alpha, \alpha^*\}$ be a CN^*-curve pair. Then, the distance function $R = R(s)$ and differential relation between curvatures are given by*

$$R(s) = \int_0^s \bar{\kappa}(s)ds, \int_0^{s^*} \sqrt{(\bar{\kappa}^*)^2 + (\bar{\tau}^*)^2}ds^* = \int_0^s \sqrt{(\bar{\tau} + R\gamma)^2 + R^2\beta^2}ds,$$

respectively.

Proof. By differentiating the parametric representation of α^* with respect to s and using the equations (5) and (7), we obtain the following system:

$$\left\{ \begin{array}{l} R' - \bar{\kappa} = 0, \\ \bar{\kappa}^* \frac{ds^*}{ds} = -(\bar{\tau} + R\gamma)\cos\theta + R\beta\sin\theta, \\ \bar{\tau}^* \frac{ds^*}{ds} = -(\bar{\tau} + R\gamma)\sin\theta - R\beta\cos\theta. \end{array} \right. \tag{8}$$

The first equation of system (8) gives $R(s) = \int_0^s \bar{\kappa}(s)ds$. Similarly, from the second and third equalities of (8) we easily have $\int_0^{s^*} \sqrt{(\bar{\kappa}^*)^2 + (\bar{\tau}^*)^2}ds^* = \int_0^s \sqrt{(\bar{\tau} + R\gamma)^2 + R^2\beta^2}ds$.

Corrolary 3.1. *Let $\{\alpha, \alpha^*\}$ be a CN^*-curve pair. Then, α is a helix if and only if $R(s)$ is a linear function given by $R(s) = \bar{\kappa}s$ where $\bar{\kappa} =$* constant.

Proof. Since the curve α is a helix, the harmonic curvature h is constant, i.e., $h = (\tau/\kappa) =$ constant. Then, for alternative curvature $\bar{\kappa}$, we have

$$\bar{\kappa} = \frac{\kappa}{\beta} = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} = \frac{1}{\sqrt{1 + h^2}} = constant. \tag{9}$$

From Theorem 3.1 and (9), we obtain $R(s) = \bar{\kappa}s$ where $\bar{\kappa} =$ constant.

Conversely, if $R(s) = \bar{\kappa}s$ is a linear function with constant factor $\bar{\kappa}$, then it is clear that $h = (\tau/\kappa) =$ constant, i.e., α is a helix.

Theorem 3.2. *Let $\{\alpha, \alpha^*\}$ be a CN^*-curve pair. The distance function R can be also given by one of the following functions:*

$$R = \frac{1}{A}\left(\bar{\kappa}^* \frac{ds^*}{ds} + \bar{\tau}\cos\theta\right), \qquad R = -\frac{1}{B}\left(\bar{\tau}^* \frac{ds^*}{ds} + \bar{\tau}\sin\theta\right), \tag{10}$$

where θ is the angle function between vector fields N, W^* and $A^2 + B^2 = \gamma^2 + \beta^2$.

Proof. Writing $A = \beta \sin\theta - \gamma \cos\theta$ and $B = \beta \cos\theta + \gamma \sin\theta$ in the second and third equalities of system (8), it follows

$$\left\{ \begin{array}{ll} \bar{\kappa}^* \frac{ds^*}{ds} & = -\bar{\tau}\cos\theta + RA, \\ \bar{\tau}^* \frac{ds^*}{ds} & = -\bar{\tau}\sin\theta - RB. \end{array} \right. \tag{11}$$

Finally, from (11) it is easy to write (10).

Theorem 3.3. *Let $\{\alpha, \alpha^*\}$ be a CN^*-curve pair. Then, α^* is a helix if and only if the ratio*

$$\frac{(\bar{\tau} + R\gamma)\sin\theta + R\beta\cos\theta}{(\bar{\tau} + R\gamma)\cos\theta - R\beta\sin\theta} \tag{12}$$

is constant.

Proof. Let $\{\alpha, \alpha^*\}$ be a CN^*-curve pair. From the second and third equations of system (8), we get

$$\frac{\bar{\tau}^*}{\bar{\kappa}^*} = \frac{(\bar{\tau} + R\gamma)\sin\theta + R\beta\cos\theta}{(\bar{\tau} + R\gamma)\cos\theta - R\beta\sin\theta}. \tag{13}$$

Since $\bar{\tau}^* = \tau^*/\beta^*$ and $\bar{\kappa}^* = \kappa^*/\beta^*$, from (13) we have $\frac{\bar{\tau}^*}{\bar{\kappa}^*} = \frac{\tau^*}{\kappa^*}$. Then, from (12) α^* is a helix if and only if

$$\frac{(\bar{\tau} + R\gamma)\sin\theta + R\beta\cos\theta}{(\bar{\tau} + R\gamma)\cos\theta - R\beta\sin\theta}$$

is constant.

Theorem 3.4. *Let $\{\alpha, \alpha^*\}$ be a CN^* curve pair. Then, the angle function and the relation between arc length parameters s and s^* are given by*

$$\theta = \int_0^{s^*} \gamma^* ds^*, s = \int_0^{s^*} \frac{\beta^*}{\gamma} \cos\theta ds^*, \tag{14}$$

respectively.

Proof. From (7), we have $W = \cos\theta C^* - \sin\theta W^*$. Differentiating this equality and using the derivative formulae in (3), we get

$$-\gamma C \frac{ds}{ds^*} = \left(\gamma^* - \frac{d\theta}{ds^*}\right) \sin\theta C^* - \beta^* \cos\theta N^* + \left(\gamma^* - \frac{d\theta}{ds^*}\right) \cos\theta W^*. \tag{15}$$

Since $\{\alpha, \alpha^*\}$ is a CN^*-curve pair, i.e., $C = N^*$, from (15) we have equalities (14).

The first equality of (14) gives us the following corollary.

Corrolary 3.2. *Let $\{\alpha, \alpha^*\}$ be a CN^*-curve pair. Then, α^* is a helix if and only if $\theta = 0$.*

Moreover, we should point out that if $\theta = 0$, then the frames of curves α and α^* coincide, i.e., α and α^* coincide. So, α is also a helix.

Theorem 3.5. *Let $\{\alpha, \alpha^*\}$ be a CN^*-curve pair. Then, α^* is a slant helix if and only if there exists a constant c which satisfies the following condition:*

$$\theta = \int_0^{s} c\left(\cos\theta\gamma - \sin\theta\beta\right) ds. \tag{16}$$

Proof. Let α^* be a slant helix. Then, from Remark 2.1 we have that $\gamma^*/\beta^* = \text{constant} = c$. Differentiating the first equality in (14), it follows

$$\gamma^* = \frac{d\theta}{ds}\frac{ds}{ds^*}. \tag{17}$$

On the other hand, since we have $N^* = C$, from the differential of this equality, we get

$$\beta^* = \frac{ds}{ds^*}\left(\cos\theta\gamma - \sin\theta\beta\right). \tag{18}$$

From (17) and (18), we have

$$\frac{\gamma^*}{\beta^*} = \frac{\frac{d\theta}{ds}}{\cos\theta\gamma - \sin\theta\beta},$$

Then we obtained that α^* is a slant helix, i.e., $\gamma^*/\beta^* = \text{constant} = c$ if and only if $\theta = \int_0^s c\left(\cos\theta\gamma - \sin\theta\beta\right)\, ds$ holds.

Theorem 3.6. *Let $\{\alpha, \alpha^*\}$ be a CN^*-curve pair and α^* be a slant helix. Then, α is a slant helix if and only if*

$$\frac{d\theta}{ds} = c\left(m\cos\theta - \sin\theta\right)\beta, \tag{19}$$

where c and m are constants.

Proof. Let α^* be a slant helix. Then, from Theorem 3.5, we have

$$\frac{d\theta}{ds} = c\left(\cos\theta\gamma - \sin\theta\beta\right). \tag{20}$$

From (20) we get

$$\frac{\gamma}{\beta} = \frac{(d\theta/ds) + c\beta\sin\theta}{c\beta\cos\theta}. \tag{21}$$

Now, let α be a slant helix, i.e., $\gamma/\beta = m = \text{constant}$. Then, from (21) we obtain

$$\frac{d\theta}{ds} = c\left(m\cos\theta - \sin\theta\right)\beta. \tag{22}$$

Conversely, let (19) holds. From Theorem 3.5, we have

$$d\theta/ds = c\left(\cos\theta\gamma - \sin\theta\beta\right) \tag{23}$$

Writing (23) in (22), if follows that $\gamma/\beta = m = \text{constant}$, i.e., α is a slant helix.

Theorem 3.7. *Let $\{\alpha, \alpha^*\}$ be a CN^*-curve pair. Then, α^* is a helix if and only if*

$$\frac{RB + \bar{\tau}\sin\theta}{RA - \bar{\tau}\cos\theta} = constant, \tag{24}$$

where $A^2 + B^2 = \gamma^2 + \beta^2$.

Proof. Let α^* be a helix. Then, we have $\tau^*/\kappa^* = \text{constant}$. Considering the system (11), we obtain

$$\frac{\tau^*}{\kappa^*} = \frac{RB + \bar{\tau}\sin\theta}{-RA + \bar{\tau}\cos\theta} = \text{constant}.$$

Then it is clear that α^* is a helix if and only if (24) holds.

Theorem 3.8. *Let $\{\alpha, \alpha^*\}$ be a CN^*-curve pair and α^* be a helix. Then, α is a slant helix if and only if*

$$\frac{RM + \frac{\bar{\tau}}{\beta}N}{RN} = \text{constant}, \tag{25}$$

where $M = n\cos\theta + \sin\theta$, $N = n\sin\theta - \cos\theta$ and n is a non-zero constant.

Proof. Since α^* is a helix, for a non-zero constant n, we can write $\tau^*/\kappa^* = 1/n = \text{constant}$. From the system (11) (or from Theorem 3.7) we get

$$\frac{\gamma}{\beta} = \frac{-R\,(n\cos\theta + \sin\theta) + (-n\sin\theta + \cos\theta)\,\frac{\bar{\tau}}{\beta}}{R\,(n\sin\theta - \cos\theta)}, \tag{26}$$

Writing $M = n\cos\theta + \sin\theta$, $N = n\sin\theta - \cos\theta$ in (26), it follows

$$\frac{\gamma}{\beta} = -\frac{RM + \frac{\bar{\tau}}{\beta}N}{RN}. \tag{27}$$

Finally, from (27) and Remark 2.1, we have that α is a slant helix if and only if the function in (27) is constant.

Theorem 3.9. *Let $\{\alpha, \alpha^*\}$ be a CN^*-curve pair and curvature centers of the curves at arbitrary points α and α^* be m and m^*, respectively. Then the distance function d_{mm^*} is not constant and given by*

$$d_{mm^*} = \sqrt{\rho^2 + (R + \rho^*)^2}, \tag{28}$$

where ρ and ρ^* are curvature radius of curves α and α^*, respectively.

Proof. Let the curvature centers of α and α^* be

$$m = \alpha(s) + \rho(s)N(s), m^* = \alpha^*(s) + \rho^*(s)N^*(s), \tag{29}$$

respectively. Since $\{\alpha, \alpha^*\}$ is a CN^*-curve pair, we have $C = N^*$. Then from Definition 3.1 and from (29), we can write

$$m^* = \alpha + (R + \rho^*)\,C, \tag{30}$$

and we obtain that $d_{mm^*} = \|m - m^*\| = \sqrt{\rho^2 + (R + \rho^*)^2}$ is not constant.

Let now consider the Mannheim theorem given for Bertrand curves. The theorem says that if α and α^* are Bertrand curves with curvature centers m and m^*, respectively, then the ratio,

$$\frac{\|\alpha^* m\|}{\|\alpha m\|} : \frac{\|\alpha^* m^*\|}{\|\alpha m^*\|} \tag{31}$$

is constant. If we compute this ratio for CN^*-curve pairs, from (29) and (30) we obtain the equalities

$$\|\alpha^* m\| = R^2 + \rho^2, \quad \|\alpha m\| = \rho, \quad \|\alpha^* m^*\| = \rho^*, \quad \|\alpha m^*\| = R + \rho^*. \tag{32}$$

Then, we have

$$\frac{\|\alpha^* m\|}{\|\alpha m\|} : \frac{\|\alpha^* m^*\|}{\|\alpha m^*\|} = \frac{(R^2 + \rho^2)(R + \rho^*)}{\rho\rho^*} \neq \text{constant}.$$

So, the following theorem is obtained:

Theorem 3.10. *Mannheim theorem is not valid for CN^*-curve pairs.*

4. Examples

In this section, we give some examples for the results obtained in Section 3.

Example 1. *(Helix) Let α_1 be a helix curve given by the parametrization*

$$\alpha_1(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right).$$

From Theorem 3.1, the distance function is obtained as $R(s) = \frac{\sqrt{2}}{2}s$. Then, the CN^*-partner curve α_1^* of α_1 is obtained as

$$\alpha_1^*(s) = \left(\cos\frac{s}{\sqrt{2}} + \frac{s}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}} - \frac{s}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right),$$

and Figure 1 shows the graphs of partner curves.

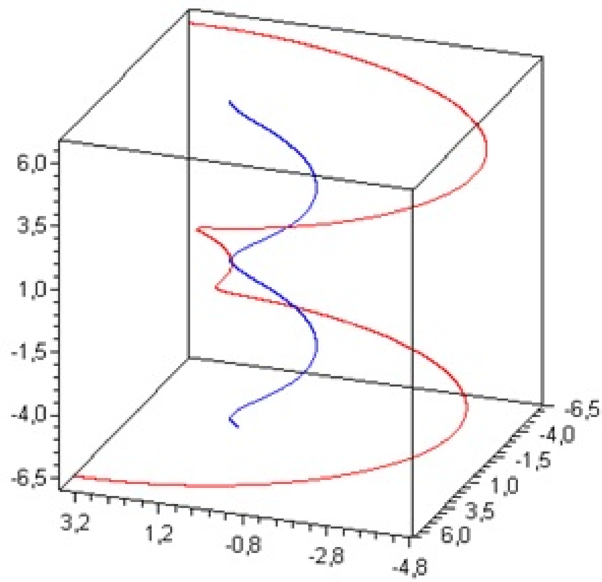

Figure 1. CN^*-partner curves α_1 (Blue) and α_1^* (Red).

Example 2. *(Slant Helix) Let the slant helix α_2 be given by the parametrization*

$$\alpha_2(s) = -\left(\frac{3}{2}\cos\left(\frac{s}{2}\right) + \frac{1}{6}\cos\left(\frac{3s}{2}\right),\ \frac{3}{2}\sin\left(\frac{s}{2}\right) + \frac{1}{6}\sin\left(\frac{3s}{2}\right),\ \sqrt{3}\cos\left(\frac{s}{2}\right)\right),$$

which is generated by a circle [5]. The distance function is obtained as $R(s) = 2\sin\left(\frac{1}{2}s\right)$. Then, the CN^*-partner curve α_2^* of α_2 is given by the parametrization

$$\alpha_2^*(s) = \left(\frac{5}{3}\cos\frac{s}{2}\left(2\cos^2\frac{s}{2} - 3\right),\ -\frac{10}{3}\sin^3\frac{s}{3},\ -\sqrt{3}\cos\frac{s}{2}\right).$$

Figure 2 shows the graph of the partner curves.

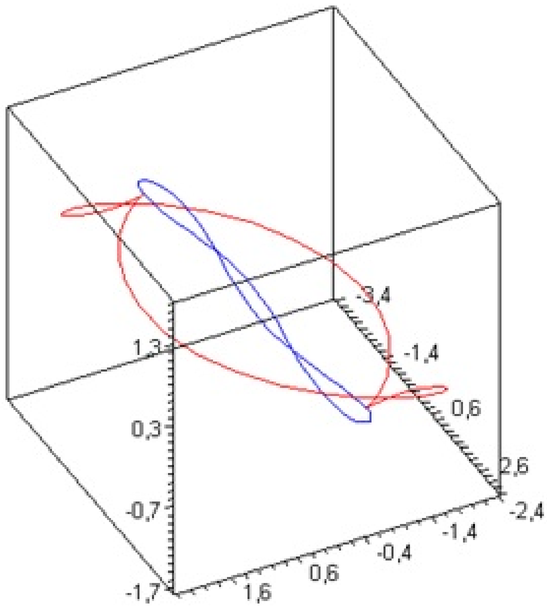

Figure 2. CN^*-partner curves α_2 (Blue) and α_2^* (Red).

Example 3. *(Circle) Let α_3 be a circle given by the parametrization*

$$\alpha_3(s) = (\cos s, \sin s, 0)\,.$$

Then, the distance function is obtained as $R(s) = s$ and CN^*-partner curve of the circle is

$$\alpha_3^*(s) = (\cos s + s \sin s,\ \ \sin s - s\cos s,\ \ 0)\,.$$

Figure 3 shows the graphs of the partner curves.

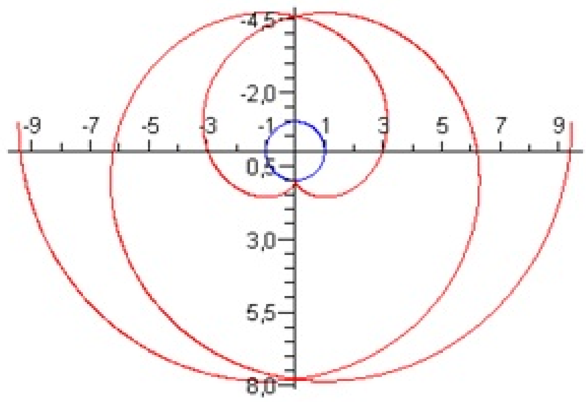

Figure 3. CN^*-partner curves α_3 (Blue) and α_3^* (Red).

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DEPARTMENT OF MATHEMATICS
MANİSA CELAL BAYAR UNIVERSITY
MURADİYE, 45140 MANİSA, TURKEY
E-mail address: `onur.kaya@cbu.edu.tr`

INDEPENDENT RESEARCHER
DELİBEKİRLİ VILLAGE, 31440, KIRIKHAN, HATAY, TURKEY
E-mail address: `mehmetonder197999@gmail.com`