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FEUERBACH'S THEOREM ON RIGHT TRIANGLE WITH AN EXTENSION

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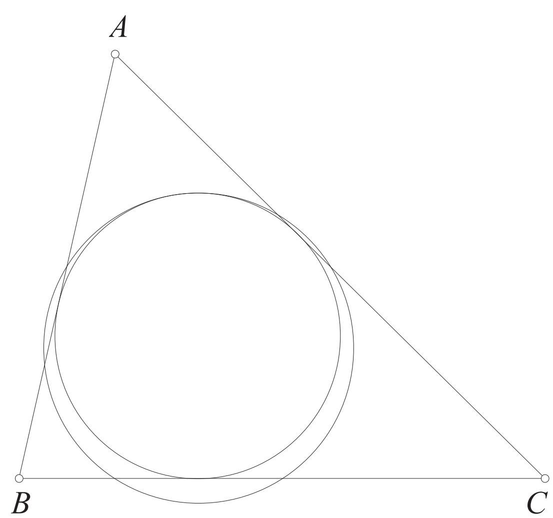
Abstract. Given a triangle, we shall construct a circle which is tangent to both the incircle and the circumcircle at a vertex of the triangle. This construction can be considered as the generalization of Feuerbach's Theorem on right triangle.

1. Introduction

Feuerbach's Theorem is known as one of the most important theorems with many applications in elementary geometry, see [1,2,3,4,5,6]

Theorem 1.1 (Feuerbach, 1822)**.** *In a nonequilateral triangle, the nine-point circle is internally tangent to the incircle.*



Recently, Michael J. G. Scheer has given a simple vector proof of this theorem [7]. For historical details of this theorem, please see [3]. In this article, we study a special case of Feuerbach's Theorem when our triangle is right.

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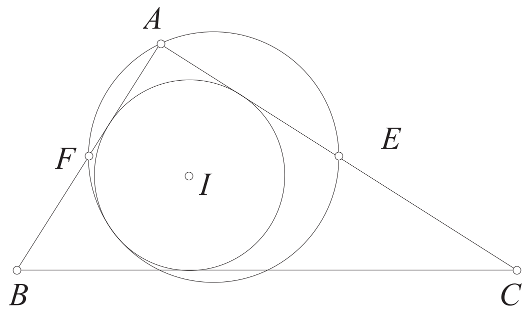
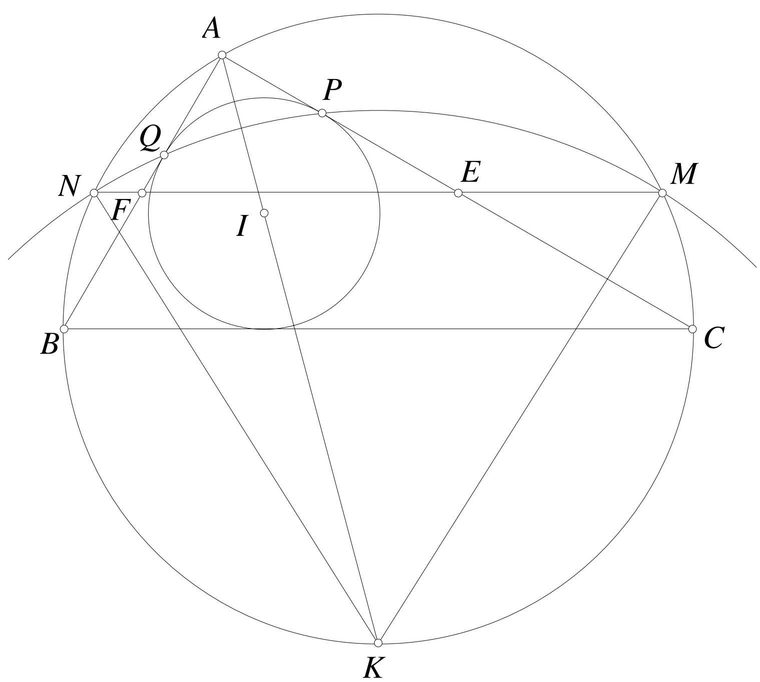
If triangle is right at a vertex, this vertex also is the feets of the altitudes from the rest vertexs, therefore the nine-point circle of this triangle passes through that vertex. We have the specical case of Feuerbach's Theorem on right triangle as following

Theorem 1.2 (Feuerbach's Theorem on right triangle)**.** *Let ABC be a right triangle at A. E, F respectively are midpoints of the segments CA, AB. Then circumcircle of the triangles AEF is tangent to the incircle of triangle ABC.*



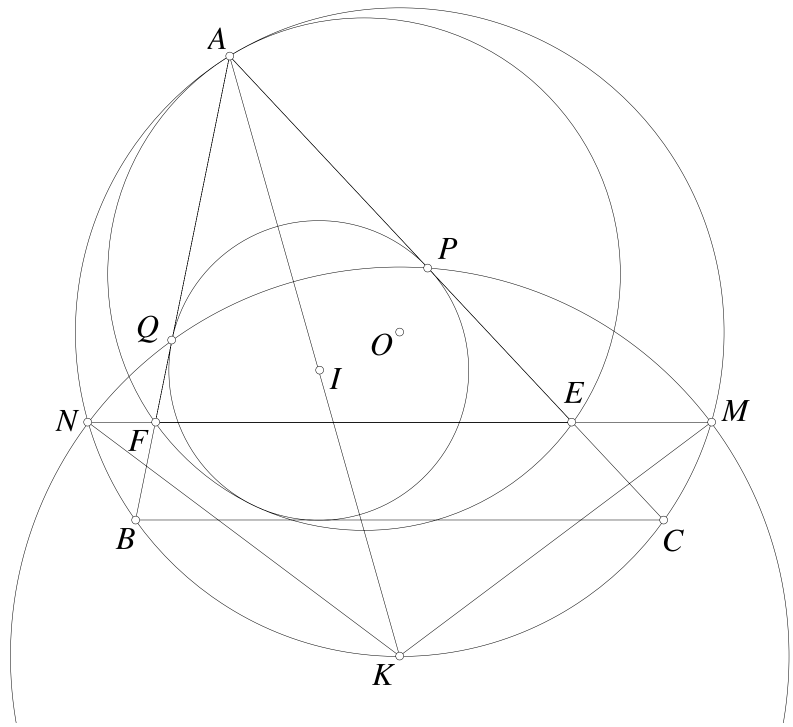
In this article, we shall present a construction of the circle that is tangent to both the incircle and the circumcircle of triangle. This can be considered as the generalization of theorem 2. First of all, we shall point out another way to construct the midline in a right triangle by the following theorem

Theorem 1.3. *Let ABC be a right triangle inscribed circle (O) with diameter BC. Incirle (I) of triangle ABC touches CA, AB at P, Q, reps. AI intersects (O) again at K. The circle (K) passes through P, Q which intersects (O) at two points M, N. Then the line MN contains the A-midline of triangle ABC.*



Now using this construction on right triangle. We establish the following interesting reuslt.

Theorem 1.4. *Let ABC be a triangle inscribed circle (O). Incirle (I) of triangle ABC touches CA, AB at P, Q, reps. AI intersects (O) again at K. The circle (K) passes through P, Q which intersects (O) at two points M, N. The line MN interects sides CA, AB at E, F, resp. Then the circumcircle of triangle AEF is tangent to (I) and (O).*



Remark 1.1. *In the theorem 1.4, if triangle ABC is right at A and we apply the construction of midline in the theorem 1.3, we get the theorem 1.2. So we see that the theorem 1.2 as a special case of the theorem 1.4.*

2. The proof of the main theorem

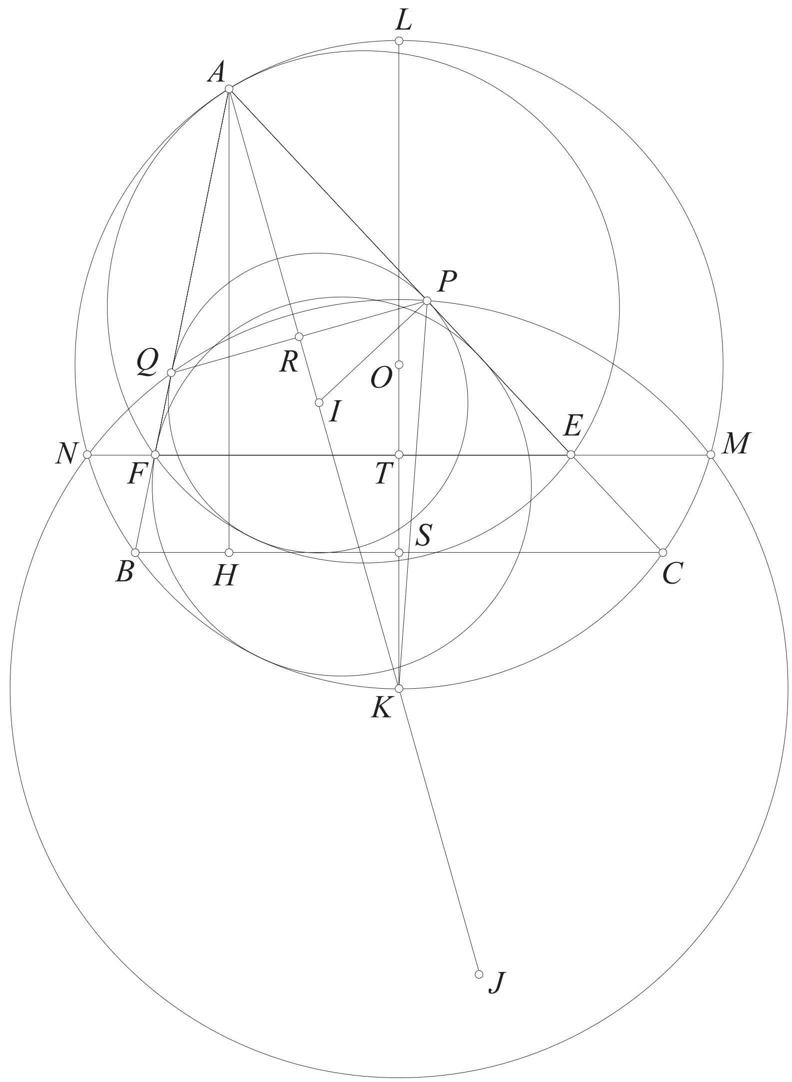
We fist present the proof of the theorem 1.4.

Proof. Let J be the A-excenter of the triangle ABC. Point H is the feet of altitude from A on line BC. The segment KL is diameter of circle (O). KL intersects BC, MN at S, T, respectively. Denote by R the midpoint of PQ. Using Pythagorean theorem and the metric relations on right triangle, we have the following indenitites

$$\begin{aligned} ST.KL &= KT \cdot KL - KS \cdot KL \\ &= KM^2 - KC^2 \\ &= KP^2 - KI^2 \\ &= KR^2 + RP^2 - KI^2 \end{aligned}$$

$$
\begin{aligned}
&= (KR - KI)(KR + KI) + RI \cdot RA \\
&= RI \cdot (KR + KJ) + RI \cdot RA \\
&= RI(RJ + RA) \\
&= RI \cdot AJ.
\end{aligned}
$$

This follows that $\frac{KL}{AJ} = \frac{RI}{ST} \quad (1)$.



Call by R the radius of the circumcircle (O). Note that, two triangles AIB and ACJ are similar. We have the indentities

$$AH \cdot KL = AH \cdot 2\mathrm{R} = \frac{2[ABC]}{BC} \cdot 2\mathrm{R} = \frac{2AB \cdot BC \cdot CA}{4\mathrm{R} \cdot BC} \cdot 2\mathrm{R} = AB \cdot AC = AI \cdot AJ.$$

Which implies $\frac{KL}{AJ} = \frac{AI}{AH} \quad (2)$.

From (1) and (2), we deduce that $\frac{RI}{ST} = \frac{AI}{AH}$. Hence,

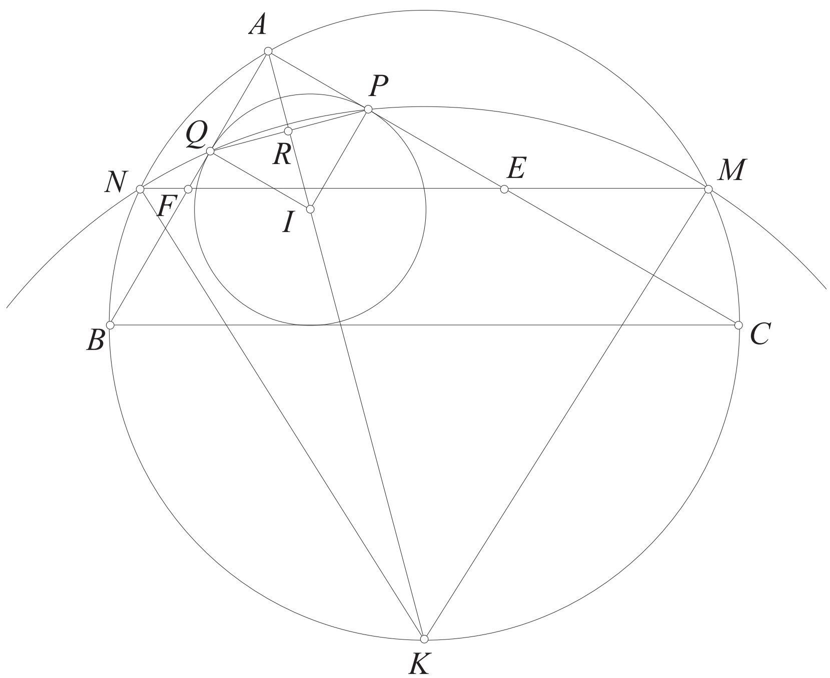
$$\frac{RI}{AI} = \frac{ST}{AH} = \frac{BF}{BA} = \frac{CE}{CA}.$$

Therefore, we have the equal ratios $\frac{AR}{AI} = \frac{AE}{AC} = \frac{AF}{AB} = k$. Consider the homothety center A with the ratio k. This homothety transforms the circumcircle of triangle AEF to the circumcircle (O) of triangle ABC, the incircle (I) transforms to A-mixtilinear incircle of ABC. Because of the touching of A-mixtilinear incircle the circumcircle (O), the circumcircle of triangle AEF and the incircle (I) of ABC are tangent.

Pretty obvious that EF is parallel to BC so the circumcircle of triangle AEF is tangent to (O) at A. We complete the proof.

Remark 2.1. *The proof of the theorem 1.4 also contains the proof of the theorem 3. Indeed, in the theorem 3, let the line MN intersect the sides CA, AB at E, F, respectively. Let the intersection of AI and PQ be the point R. Because triangle ABC is right at A, R is midpoint of AI. Using the last equal ratios in the proof of the theorem 1.4, we have*

$$\frac{AE}{AC} = \frac{AF}{AB} = \frac{AR}{AI} = \frac{1}{2}.$$



Thus E, F are midpoints of CA, AB. This means the line MN contains the A-midline of triangle ABC.

3. Conclusion

In this article, we have reiterated the famous theorem of Feuerbach. Then we apply the Feuerbach's theorem on the right triangle. Using a new construction of midline in the right triangle, we extended Feuerbach's theorem on the right triangle and gave a purely geometric solution.

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