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AN n-DIMENSIONAL GENERALIZATION OF A GEOMETRIC INEQUALITY

Vandanjav Adiyasuren and Tsetsendelger Taivansaikhan

Abstract. In this paper, we shall give an n-dimensional generalization of a geometric inequality posed by Nguyen Viet Hung [1].

1. Introduction

Nguyen Viet Hung [1] proposed following problem: Let ABC be a triangle and G be its centroid. Lines AG, BG, CG meet the circumcircle of triangle ABC at A_1, B_1, C_1 respectively. Prove that

$$\sqrt{a^2+b^2+c^2} \leq GA_1 + GB_1 + GC_1 \leq 2R + \frac{1}{6}\left(\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c}\right).$$

In this paper, we shall give an n-dimensional generalization of the inequality.

2. Main result

In order to prove the main theorem, we need the following three lemmas.

Lemma 2.1. *(See [2].) Let m_1 be the median of $A_1 \dots A_n A_{n+1}, (n \geq 2)$ from vertex A_1. Then*

$$m_1^2 = \frac{1}{n^2}\left(n \sum_{2\leq j\leq n+1} A_1A_j^2 - \sum_{2\leq i<j\leq n+1} A_iA_j^2\right). \tag{1}$$

Lemma 2.2. *(See [2].) Let an n-dimensional simplex $A_1 \dots A_n A_{n+1}$ with centroid G be inscribed in a sphere of radius R. The line A_1G meet the sphere again at A_1'. Then, we have*

$$m_1 \cdot G_1A_1' = \frac{1}{n^2} \sum_{2\leq i<j\leq n+1} A_iA_j^2. \tag{2}$$

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Lemma 2.3.

$$\sum_{1\le k\le n+1} m_k^2 = \frac{n+1}{n^2}\sum_{1\le i<j\le n+1} A_iA_j^2. \tag{3}$$

Proof. Conclusion the Lemma 2.1.

The main result of this section is the following theorem.

Theorem 2.1. *Let an n-dimensional simplex $A_1\dots A_nA_{n+1}$ with centroid G be inscribed in a sphere of radius R. The lines $A_1G,\dots,A_{n+1}G$ meet the sphere again $A_1',\dots,A_{n+1}'$ respectively. Then, we have*

$$\begin{aligned}\sqrt{\sum_{1\le i<j\le n+1} A_iA_j^2} &\le \sum_{1\le\ j\le n+1} GA_j' \\ &\le 2R+\frac{1}{n(n+1)}\left(\frac{1}{m_1}\sum_{2\le i<j\le n+1} A_iA_j^2+\dots+\frac{1}{m_{n+1}}\sum_{1\le i<j\le n} A_iA_j^2\right)\end{aligned} \tag{4}$$

Proof. Let us prove the right-hand side inequality in (1), using Lemma 2.2, we have

$$2R \ge A_1A_1' = A_1G_1 + G_1A_1' = m_1 + \frac{1}{n^2}\cdot\frac{1}{m_1}\cdot\sum_{2\le i<j\le n+1} A_iA_j^2,$$

where G_1 denote the centroid of simplex $A_2A_3\dots A_{n+1}$. Then we have

$$2n^2R \le n^2\cdot m_1 + \frac{1}{m_1}\cdot\sum_{2\le i<j\le n+1} A_iA_j^2.$$

By the Lemma 2.1, we get

$$\begin{aligned}GA_1' &= GG_1 + G_1A_1' \\ &= \frac{1}{n+1}\cdot m_1 + \frac{1}{n^2}\cdot\frac{1}{m_1}\cdot\sum_{2\le i<j\le n+1} A_iA_j^2 \\ &= \frac{1}{n^2(n+1)}\left(n^2m_1+\frac{1}{m_1}\sum_{2\le i<j\le n+1} A_iA_j^2\right) + \frac{1}{n(n+1)}\cdot\frac{1}{m_1}\cdot\sum_{2\le i<j\le n+1} A_iA_j^2 \\ &\le \frac{1}{n^2(n+1)}\cdot 2n^2R + \frac{1}{n(n+1)}\cdot\frac{1}{m_1}\cdot\sum_{2\le i<j\le n+1} A_iA_j^2 \\ &= \frac{2R}{n+1} + \frac{1}{n(n+1)}\cdot\frac{1}{m_1}\cdot\sum_{2\le i<j\le n+1} A_iA_j^2.\end{aligned}$$

We can prove similar inequalities for $m_i, i = 2,\dots,n+1$. Then, summing up these inequalities, we get

$$\sum_{1\le j\le n+1} GA_j' \le 2R+\frac{1}{n(n+1)}\left(\frac{1}{m_1}\sum_{2\le i<j\le n+1} A_iA_j^2+\dots+\frac{1}{m_{n+1}}\sum_{1\le i<j\le n} A_iA_j^2\right).$$

Next we prove the left-hand side inequality. Using following formula and Lemma 2.1, we get

$$\begin{aligned}
GA_1 &= GG_1 + G_1A_1' \\
&= \frac{1}{n+1}\cdot m_1 + \frac{1}{n^2}\cdot\frac{1}{m_1}\cdot\sum_{2\le i<j\le n+1} A_iA_j^2 \\
&= \frac{1}{m_1}\left(\frac{1}{n+1}\cdot m_1^2 + \frac{1}{n^2}\cdot\sum_{2\le i<j\le n+1} A_iA_j^2\right) \\
&= \frac{1}{m_1}\left(\frac{1}{n(n+1)}\cdot\sum_{2\le j\le n+1} A_1A_j^2 - \frac{1}{n^2(n+1)}\cdot\sum_{2\le i<j\le n+1} A_iA_j^2 + \frac{1}{n^2}\cdot\sum_{2\le i<j\le n+1} A_iA_j^2\right) \\
&= \frac{1}{m_1}\cdot\frac{1}{n(n+1)}\cdot\sum_{1\le i<j\le n+1} A_iA_j^2.
\end{aligned}$$

hence we have

$$\sum_{1\le j\le n+1} GA_j' = \frac{1}{n(n+1)}\left(\sum_{1\le i<j\le n+1} A_iA_j^2\right)\left(\sum_{1\le k\le n+1}\frac{1}{m_k}\right).$$

Applying AM-GM inequality two times and Cauchy-Schwartz's inequality and Lemma 2.3, we have

$$\begin{aligned}
\sum_{1\le j\le n+1} GA_j' &\ge \frac{1}{n(n+1)}\cdot\left(\sum_{1\le i<j\le n+1} A_iA_j^2\right)\cdot\frac{n+1}{\sqrt[n+1]{\prod_{1\le k\le n+1} m_k}} \\
&\ge \frac{n+1}{n}\cdot\left(\sum_{1\le i<j\le n+1} A_iA_j^2\right)\cdot\frac{1}{\sum_{1\le k\le n+1} m_k} \\
&\ge \frac{n+1}{n}\cdot\left(\sum_{1\le i<j\le n+1} A_iA_j^2\right)\cdot\frac{1}{\sqrt{(n+1)}\cdot\sqrt{\prod_{1\le k\le n+1} m_k^2}} \\
&= \sqrt{\sum_{1\le i<j\le n+1} A_iA_j^2}.
\end{aligned}$$

Which completes the proof.

If we take $n = 2$ in inequality (4), we get the following corollary.

Corollary 2.1. *Let a triangle $A_1A_2A_3$ with the center G be inscribed in a circle of radius R. The lines A_1G, A_2G, A_3G meet the circle again at A_1', A_2', A_3' respectively. Then*

$$\sqrt{A_1A_2^2 + A_2A_3^2 + A_3A_1^2} \le GA_1' + GA_2' + GA_3' \le 2R + \frac{1}{6}\left(\frac{A_2A_3^2}{m_1} + \frac{A_1A_3^2}{m_2} + \frac{A_1A_2^2}{m_3}\right).$$

References

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DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY MONGOLIA
ULAANBAATAR, MONGOLIA
E-mail address: V_Adiyasuren@yahoo.com

DEPARTMENT OF MATHEMATICS
NEW MONGOL INSTITUTE OF TECHNOLOGY
ULAANBAATAR, MONGOLIA
E-mail address: ts.taivansaikhan@gmail.com