



**MERIDIAN SURFACES WITH PARALLEL
NORMALIZED MEAN CURVATURE VECTOR FIELD IN
PSEUDO-EUCLIDEAN 4-SPACE WITH NEUTRAL METRIC**

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Abstract. We construct a special class of Lorentz surfaces in the pseudo-Euclidean 4-space with neutral metric which are one-parameter systems of meridians of rotational hypersurfaces with timelike or spacelike axis and call them meridian surfaces. We give the complete classification of the meridian surfaces with parallel mean curvature vector field. We also classify the meridian surfaces with parallel normalized mean curvature vector. We show that in the family of the meridian surfaces there exist Lorentz surfaces which have parallel normalized mean curvature vector field but not parallel mean curvature vector.

1. INTRODUCTION

A basic class of surfaces in Riemannian and pseudo-Riemannian geometry are surfaces with parallel mean curvature vector field, since they are critical points of some natural functionals and play important role in differential geometry, the theory of harmonic maps, as well as in physics. The classification of surfaces with parallel mean curvature vector field in Riemannian space forms was given by Chen [4] and Yau [18]. Recently, spacelike surfaces with parallel mean curvature vector field in pseudo-Euclidean spaces with arbitrary codimension were classified in [6] and [7]. Lorentz surfaces with parallel mean curvature vector field in arbitrary pseudo-Euclidean space \mathbb{E}_s^m are studied in [8] and [11]. A nice survey on classical and recent results on submanifolds with parallel mean curvature vector in Riemannian manifolds as well as in pseudo-Riemannian manifolds is presented in [9].

The class of surfaces with parallel mean curvature vector field is naturally extended to the class of surfaces with parallel normalized mean curvature vector field.

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A submanifold in a Riemannian manifold is said to have parallel normalized mean curvature vector field if the mean curvature vector is non-zero and the unit vector in the direction of the mean curvature vector is parallel in the normal bundle [5]. It is well known that submanifolds with non-zero parallel mean curvature vector field also have parallel normalized mean curvature vector field. But the condition to have parallel normalized mean curvature vector field is weaker than the condition to have parallel mean curvature vector field. Every surface in the Euclidean 3-space has parallel normalized mean curvature vector field but in the 4-dimensional Euclidean space, there exist abundant examples of surfaces which lie fully in \mathbb{E}^4 with parallel normalized mean curvature vector field, but not with parallel mean curvature vector field. In [5] it is proved that every analytic surface with parallel normalized mean curvature vector in the Euclidean space \mathbb{E}^m must either lie in a 4-dimensional space \mathbb{E}^4 or in a hypersphere of \mathbb{E}^m as a minimal surface.

In the pseudo-Euclidean space with neutral metric \mathbb{E}_2^4 the study of Lorentz surfaces with parallel normalized mean curvature vector field, but not parallel mean curvature vector field, is still an open problem.

In the present paper we construct special families of 2-dimensional Lorentz surfaces in \mathbb{E}_2^4 which lie on rotational hypersurfaces with timelike or spacelike axis and call them meridian surfaces. Depending on the type of the spheres in \mathbb{E}_1^3 (resp. \mathbb{E}_2^3) and the casual character of the spherical curves, we distinguish three types of Lorentz meridian surfaces in \mathbb{E}_2^4 . These surfaces are analogous to the meridian surfaces in the Euclidean space \mathbb{E}^4 and the Minkowski space \mathbb{E}_1^4 , which are defined and studied in [12], [14], and [13], [15], respectively.

In Theorems 4.1, 4.2, and 4.3 we give the complete classification of all Lorentz meridian surfaces (of these three types) which have parallel mean curvature vector field. We also classify the meridian surfaces with parallel normalized mean curvature vector field (Theorems 5.1, 5.2, and 5.3). In the family of the meridian surfaces we find examples of Lorentz surfaces in \mathbb{E}_2^4 which have parallel normalized mean curvature vector field but not parallel mean curvature vector field.

2. PRELIMINARIES

Let \mathbb{E}_2^4 be the pseudo-Euclidean 4-dimensional space with the canonical pseudo-Euclidean metric of index 2 given in local coordinates by

$$\tilde{g} = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of \mathbb{E}_2^4 . We denote by $\langle \cdot, \cdot \rangle$ the indefinite inner scalar product with respect to \tilde{g} . Since \tilde{g} is an indefinite metric, a vector $v \in \mathbb{E}_2^4$ can have one of the three casual characters: it can be *spacelike* if $\langle v, v \rangle > 0$ or $v = 0$, *timelike* if $\langle v, v \rangle < 0$, and *null* (*lightlike*) if $\langle v, v \rangle = 0$ and $v \neq 0$.

We use the following denotations:

$$\begin{aligned} \mathbb{S}_2^3(1) &= \{V \in \mathbb{E}_2^4 : \langle V, V \rangle = 1\}; \\ \mathbb{H}_1^3(-1) &= \{V \in \mathbb{E}_2^4 : \langle V, V \rangle = -1\}. \end{aligned}$$

The space $\mathbb{S}_2^3(1)$ is known as the de Sitter space, and the space $\mathbb{H}_1^3(-1)$ is the hyperbolic space (or the anti-de Sitter space) [16].

Given a surface M in \mathbb{E}_2^4 , we denote by g the induced metric of \tilde{g} on M . A surface M in \mathbb{E}_2^4 is called *Lorentz* if the induced metric g on M is Lorentzian. Thus, at each point $p \in M$ we have the following decomposition

$$\mathbb{E}_2^4 = T_p M \oplus N_p M$$

with the property that the restriction of the metric onto the tangent space $T_p M$ is of signature $(1, 1)$, and the restriction of the metric onto the normal space $N_p M$ is of signature $(1, 1)$.

Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and \mathbb{E}_2^4 , respectively. For any vector fields x, y tangent to M the Gauss formula is given by

$$\tilde{\nabla}_x y = \nabla_x y + h(x, y),$$

where h is the second fundamental form of M . Let D denotes the normal connection on the normal bundle of M . Then for any normal vector field ξ and any tangent vector field x the Weingarten formula is given by

$$\tilde{\nabla}_x \xi = -A_\xi x + D_x \xi,$$

where A_ξ is the shape operator with respect to ξ .

The mean curvature vector field H of M in \mathbb{E}_2^4 is defined as $H = \frac{1}{2} \text{tr } h$. A surface M is called *minimal* if its mean curvature vector vanishes identically, i.e. $H = 0$. A natural extension of minimal surfaces are quasi-minimal surfaces. A surface M is called *quasi-minimal* (or *pseudo-minimal*) if its mean curvature vector is lightlike at each point, i.e. $H \neq 0$ and $\langle H, H \rangle = 0$ [17].

A normal vector field ξ on M is called *parallel in the normal bundle* (or simply *parallel*) if $D\xi = 0$ holds identically [10]. A surface M is said to have *parallel mean curvature vector field* if its mean curvature vector H satisfies $DH = 0$ identically.

Surfaces for which the mean curvature vector field H is non-zero, $\langle H, H \rangle \neq 0$, and there exists a unit vector field b in the direction of the mean curvature vector H , such that b is parallel in the normal bundle, are called surfaces with *parallel normalized mean curvature vector field* [5]. It is easy to see that if M is a surface with non-zero parallel mean curvature vector field H (i.e. $DH = 0$), then M is a surface with parallel normalized mean curvature vector field, but the converse is not true in general. It is true only in the case $\|H\| = \text{const}$.

3. CONSTRUCTION OF MERIDIAN SURFACES IN \mathbb{E}_2^4

In [12] G. Ganchev and the second author constructed a family of surfaces lying on a standard rotational hypersurface in the Euclidean 4-space \mathbb{E}^4 . These surfaces are one-parameter systems of meridians of the rotational hypersurface, that is why they called them *meridian surfaces*. In [12] and [14] they gave the classification of the meridian surfaces with constant Gauss curvature, with constant mean curvature, Chen meridian surfaces and meridian surfaces with parallel normal bundle. The meridian surfaces in \mathbb{E}^4 with pointwise 1-type Gauss map are classified in [1]. In [13] and [15] they

used the idea from the Euclidean case to construct special families of two-dimensional spacelike surfaces lying on rotational hypersurfaces in \mathbb{E}_1^4 with timelike or spacelike axis and gave the classification of meridian surfaces from the same basic classes. The meridian surfaces in \mathbb{E}_1^4 with pointwise 1-type Gauss map are classified in [2].

Following the idea from the Euclidean and Minkowski spaces, in [3] we constructed Lorentz meridian surfaces in the pseudo-Euclidean 4-space \mathbb{E}_2^4 as one-parameter systems of meridians of rotational hypersurfaces with timelike or spacelike axis. We gave the classification of quasi-minimal meridian surfaces and meridian surfaces with constant mean curvature (CMC-surfaces). Here we shall present briefly the construction.

3.1. Lorentz meridian surfaces lying on a rotational hypersurface with timelike axis.

Let $Oe_1e_2e_3e_4$ be a fixed orthonormal coordinate system in \mathbb{E}_2^4 , i.e. $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1, \langle e_3, e_3 \rangle = \langle e_4, e_4 \rangle = -1$. We shall consider a rotational hypersurface with timelike axis Oe_4 . Similarly, one can consider a rotational hypersurface with axis Oe_3 .

In the Minkowski space $\mathbb{E}_1^3 = \text{span}\{e_1, e_2, e_3\}$ there are two types of two-dimensional spheres, namely the pseudo-sphere $\mathbb{S}_1^2(1) = \{V \in \mathbb{E}_1^3 : \langle V, V \rangle = 1\}$ and the pseudo-hyperbolic space $\mathbb{H}^2(-1) = \{V \in \mathbb{E}_1^3 : \langle V, V \rangle = -1, V^3 > 0\}$, where V^3 is the third coordinate of V . So, we can consider two types of rotational hypersurfaces about the axis Oe_4 .

Let $f = f(u), g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$. The first type rotational hypersurface \mathcal{M}^I in \mathbb{E}_2^4 , obtained by the rotation of the meridian curve $m : u \rightarrow (f(u), g(u))$ about the Oe_4 -axis, is parametrized as follows:

$$\mathcal{M}^I : Z(u, w^1, w^2) = f(u) (\cosh w^1 \cos w^2 e_1 + \cosh w^1 \sin w^2 e_2 + \sinh w^1 e_3) + g(u) e_4.$$

Note that $l^I(w^1, w^2) = \cosh w^1 \cos w^2 e_1 + \cosh w^1 \sin w^2 e_2 + \sinh w^1 e_3$ is the unit position vector of the sphere $\mathbb{S}_1^2(1)$ in $\mathbb{E}_1^3 = \text{span}\{e_1, e_2, e_3\}$ centered at the origin O . The parametrization of \mathcal{M}^I can be written as:

$$\mathcal{M}^I : Z(u, w^1, w^2) = f(u)l^I(w^1, w^2) + g(u) e_4.$$

The second type rotational hypersurface \mathcal{M}^{II} in \mathbb{E}_2^4 , obtained by the rotation of the meridian curve m about the axis Oe_4 , is given by the following parametrization:

$$\mathcal{M}^{II} : Z(u, w^1, w^2) = f(u) (\sinh w^1 \cos w^2 e_1 + \sinh w^1 \sin w^2 e_2 + \cosh w^1 e_3) + g(u) e_4.$$

If we denote by $l^{II}(w^1, w^2) = \sinh w^1 \cos w^2 e_1 + \sinh w^1 \sin w^2 e_2 + \cosh w^1 e_3$ the unit position vector of the hyperbolic sphere $\mathbb{H}_1^2(-1)$ in $\mathbb{E}_1^3 = \text{span}\{e_1, e_2, e_3\}$ centered at the origin O , then the parametrization of \mathcal{M}^{II} can be written as:

$$\mathcal{M}^{II} : Z(u, w^1, w^2) = f(u)l^{II}(w^1, w^2) + g(u) e_4.$$

We shall construct Lorentz surfaces in \mathbb{E}_2^4 which are one-parameter systems of meridians of the hypersurface \mathcal{M}^I or \mathcal{M}^{II} .

Meridian surfaces on \mathcal{M}^I :

Let $w^1 = w^1(v), w^2 = w^2(v), v \in J, J \subset \mathbb{R}$. Then $c : l = l(v) = l^I(w^1(v), w^2(v))$ is a smooth curve on $\mathbb{S}_1^2(1)$. We consider the two-dimensional

surface \mathcal{M}' lying on \mathcal{M}^I and defined by:

$$(1) \quad \mathcal{M}' : z(u, v) = f(u)l(v) + g(u)e_4, \quad u \in I, v \in J.$$

The surface \mathcal{M}' , defined by (1), is a one-parameter system of meridians of \mathcal{M}^I , so we call it a *meridian surface on \mathcal{M}^I* .

The tangent space of \mathcal{M}' is spanned by the vector fields

$$z_u = f'(u)l(v) + g'(u)e_4; \quad z_v = f(u)l'(v),$$

so, the coefficients of the first fundamental form of \mathcal{M}' are

$$E = \langle z_u, z_u \rangle = f'^2 - g'^2; \quad F = \langle z_u, z_v \rangle = 0; \quad G = \langle z_v, z_v \rangle = f^2 \langle l', l' \rangle.$$

Since we are interested in Lorentz surfaces, in the case the spherical curve c is spacelike, i.e. $\langle l', l' \rangle > 0$, we take the meridian curve m to be timelike, i.e. $f'^2 - g'^2 < 0$; and if c is timelike, i.e. $\langle l', l' \rangle < 0$, we take m to be spacelike, i.e. $f'^2 - g'^2 > 0$.

Case (a): Let $\langle l', l' \rangle = 1$, i.e. c is spacelike. We denote by $t(v) = l'(v)$ the tangent vector field of c . Since $\langle t(v), t(v) \rangle = 1$, $\langle l(v), l(v) \rangle = 1$, and $\langle t(v), l(v) \rangle = 0$, there exists a unique (up to a sign) vector field $n(v)$, such that $\{l(v), t(v), n(v)\}$ is an orthonormal frame field in \mathbb{E}_1^3 (note that $\langle n(v), n(v) \rangle = -1$). With respect to this frame field we have the following Frenet formulas of c on $\mathbb{S}_1^2(1)$:

$$(2) \quad \begin{aligned} l' &= t; \\ t' &= -\kappa n - l; \\ n' &= -\kappa t, \end{aligned}$$

where $\kappa(v) = \langle t'(v), n(v) \rangle$ is the spherical curvature of c on $\mathbb{S}_1^2(1)$.

Without loss of generality we assume that $f'^2 - g'^2 = -1$. Then for the coefficients of the first fundamental form we have $E = -1$; $F = 0$; $G = f^2(u)$. Hence, in this case the meridian surface, defined by (1), is a Lorentz surface in \mathbb{E}_2^4 . We denote this surface by \mathcal{M}'_a .

Now we consider the unit tangent vector fields $X = z_u$, $Y = \frac{z_v}{f} = t$, which satisfy $\langle X, X \rangle = -1$, $\langle Y, Y \rangle = 1$ and $\langle X, Y \rangle = 0$, and the following normal vector fields:

$$(3) \quad n_1 = n(v); \quad n_2 = g'(u)l(v) + f'(u)e_4.$$

Thus we obtain a frame field $\{X, Y, n_1, n_2\}$ of \mathcal{M}'_a , such that $\langle n_1, n_1 \rangle = -1$, $\langle n_2, n_2 \rangle = 1$, $\langle n_1, n_2 \rangle = 0$.

Taking into account (2) we get:

$$(4) \quad \begin{aligned} h(X, X) &= \kappa_m n_2; \\ h(X, Y) &= 0; \\ h(Y, Y) &= -\frac{\kappa}{f} n_1 - \frac{g'}{f} n_2, \end{aligned}$$

where κ_m denotes the curvature of the meridian curve m , i.e. $\kappa_m(u) = f''g' - f'g''$. Formulas (4) and the equality $f'^2 - g'^2 = -1$ imply that the

Gauss curvature K and the normal mean curvature vector field H of the meridian surface \mathcal{M}'_a are given, respectively by

$$(5) \quad K = \frac{f''}{f};$$

$$(6) \quad H = -\frac{\kappa}{2f} n_1 - \frac{f f'' + f'^2 + 1}{2f \sqrt{f'^2 + 1}} n_2.$$

Case (b): Let $\langle l', l' \rangle = -1$, i.e. c is timelike. In this case we assume that $f'^2 - g'^2 = 1$. We denote $t(v) = l'(v)$ and consider an orthonormal frame field $\{l(v), t(v), n(v)\}$ of \mathbb{E}_1^3 , such that $\langle l, l \rangle = 1$, $\langle t, t \rangle = -1$, $\langle n, n \rangle = 1$. Then we have the following Frenet formulas of c on $\mathbb{S}_1^2(1)$:

$$(7) \quad \begin{aligned} l' &= t; \\ t' &= \kappa n + l; \\ n' &= \kappa t, \end{aligned}$$

where $\kappa(v) = \langle t'(v), n(v) \rangle$ is the spherical curvature of c on $\mathbb{S}_1^2(1)$.

In this case the coefficients of the first fundamental form are $E = 1$; $F = 0$; $G = -f^2(u)$. We denote the meridian surface in this case by \mathcal{M}'_b .

Again we consider the unit tangent vector fields $X = z_u$, $Y = \frac{z_v}{f} = t$, which satisfy $\langle X, X \rangle = 1$, $\langle Y, Y \rangle = -1$ and $\langle X, Y \rangle = 0$, and the following normal vector fields:

$$n_1 = n(v); \quad n_2 = g'(u) l(v) + f'(u) e_4,$$

satisfying $\langle n_1, n_1 \rangle = 1$, $\langle n_2, n_2 \rangle = -1$, $\langle n_1, n_2 \rangle = 0$.

Using (7) we get:

$$(8) \quad \begin{aligned} h(X, X) &= \kappa_m n_2; \\ h(X, Y) &= 0; \\ h(Y, Y) &= \frac{\kappa}{f} n_1 - \frac{g'}{f} n_2, \end{aligned}$$

where κ_m is the curvature of the meridian curve m , which in the case of a spacelike curve is given by the formula $\kappa_m(u) = f'g'' - f''g'$. Formulas (8) and the equality $f'^2 - g'^2 = 1$ imply that the Gauss curvature K and the normal mean curvature vector field H of the meridian surface \mathcal{M}'_b are expressed as follows:

$$(9) \quad \begin{aligned} K &= -\frac{f''}{f}; \\ H &= -\frac{\kappa}{2f} n_1 + \frac{f f'' + f'^2 - 1}{2f \sqrt{f'^2 - 1}} n_2. \end{aligned}$$

Meridian surfaces on \mathcal{M}^{II} :

Now we shall construct meridian surfaces lying on the rotational hypersurface of second type \mathcal{M}^{II} . Let $c : l = l(v) = l^{II}(w^1(v), w^2(v))$ be a smooth curve on the hyperbolic sphere $\mathbb{H}^2(-1)$, where $w^1 = w^1(v)$,

$w^2 = w^2(v)$, $v \in J$, $J \subset \mathbb{R}$. We consider the two-dimensional surface \mathcal{M}'' lying on \mathcal{M}^{II} and defined by:

$$(10) \quad \mathcal{M}'' : z(u, v) = f(u)l(v) + g(u)e_4, \quad u \in I, v \in J.$$

The surface \mathcal{M}'' , defined by (10), is a one-parameter system of meridians of \mathcal{M}^{II} , so we call it a *meridian surface on \mathcal{M}^{II}* .

The tangent space of \mathcal{M}'' is spanned by the vector fields

$$z_u = f'(u)l(v) + g'(u)e_4; \quad z_v = f(u)l'(v)$$

and the coefficients of the first fundamental form of \mathcal{M}'' are

$$E = \langle z_u, z_u \rangle = -(f'^2 + g'^2); \quad F = \langle z_u, z_v \rangle = 0; \quad G = \langle z_v, z_v \rangle = f^2 \langle l', l' \rangle.$$

Since c is a curve lying on $\mathbb{H}^2(-1)$, we have $\langle l, l \rangle = -1$, so $t = l'$ satisfies $\langle t, t \rangle = 1$. We suppose that $f'^2 + g'^2 = 1$. Hence, the coefficients of the first fundamental form of \mathcal{M}'' are $E = -1$; $F = 0$; $G = f^2$.

Now, we have an orthonormal frame field $\{l(v), t(v), n(v)\}$ of c satisfying the conditions $\langle l, l \rangle = -1$, $\langle t, t \rangle = 1$, $\langle n, n \rangle = 1$, and the following Frenet formulas of c on $\mathbb{H}^2(-1)$ hold true:

$$(11) \quad \begin{aligned} l' &= t; \\ t' &= \kappa n + l; \\ n' &= -\kappa t, \end{aligned}$$

where $\kappa(v) = \langle t'(v), n(v) \rangle$ is the spherical curvature of c on $\mathbb{H}^2(-1)$.

We consider the following orthonormal frame field of \mathcal{M}'' :

$$X = z_u; \quad Y = \frac{z_v}{f} = t; \quad n_1 = n(v); \quad n_2 = -g'(u)l(v) + f'(u)e_4.$$

This frame field satisfies $\langle X, X \rangle = -1$, $\langle Y, Y \rangle = 1$, $\langle X, Y \rangle = 0$, $\langle n_1, n_1 \rangle = 1$, $\langle n_2, n_2 \rangle = -1$, $\langle n_1, n_2 \rangle = 0$. Using (11) we get:

$$\begin{aligned} h(X, X) &= \kappa_m n_2; \\ h(X, Y) &= 0; \\ h(Y, Y) &= \frac{\kappa}{f} n_1 - \frac{g'}{f} n_2, \end{aligned}$$

where $\kappa_m = f'g'' - f''g'$. The Gauss curvature K and the normal mean curvature vector field H of \mathcal{M}'' are given, respectively by

$$(12) \quad \begin{aligned} K &= \frac{f''}{f}; \\ H &= \frac{\kappa}{2f} n_1 + \frac{ff'' + f'^2 - 1}{2f\sqrt{1 - f'^2}} n_2. \end{aligned}$$

Note that on the rotational hypersurface \mathcal{M}^I we can consider two types of Lorentz meridian surfaces, namely surfaces of type \mathcal{M}'_a and \mathcal{M}'_b , while on \mathcal{M}^{II} we can construct only one type of Lorentz meridian surfaces, namely \mathcal{M}'' .

3.2. Lorentz meridian surfaces lying on a rotational hypersurface with spacelike axis. In this subsection we shall explain the construction of meridian surfaces lying on a rotational hypersurface with spacelike axis Oe_1 . Similarly, we can consider meridian surfaces lying on a rotational hypersurface with axis Oe_2 .

Let \mathbb{E}_2^3 be the Minkowski space $\mathbb{E}_2^3 = \text{span}\{e_2, e_3, e_4\}$. In \mathbb{E}_2^3 there are two types of spheres, namely the de Sitter space $\mathbb{S}_2^2(1) = \{V \in \mathbb{E}_2^3 : \langle V, V \rangle = 1\}$, and the hyperbolic space $\mathbb{H}_1^2(-1) = \{V \in \mathbb{E}_2^3 : \langle V, V \rangle = -1\}$. So, we can consider two types of rotational hypersurfaces about the axis Oe_1 .

Let $f = f(u)$, $g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, and $\tilde{l}^I(w^1, w^2) = \cosh w^1 e_2 + \sinh w^1 \cos w^2 e_3 + \sinh w^1 \sin w^2 e_4$ be the unit position vector of the sphere $\mathbb{S}_2^2(1)$ in $\mathbb{E}_2^3 = \text{span}\{e_2, e_3, e_4\}$ centered at the origin O . The first type rotational hypersurface $\widetilde{\mathcal{M}}^I$, obtained by the rotation of the meridian curve $m : u \rightarrow (f(u), g(u))$ about the axis Oe_1 , is parametrized as follows:

$\widetilde{\mathcal{M}}^I : Z(u, w^1, w^2) = g(u) e_1 + f(u) (\cosh w^1 e_2 + \sinh w^1 \cos w^2 e_3 + \sinh w^1 \sin w^2 e_4)$,
or equivalently,

$$\widetilde{\mathcal{M}}^I : Z(u, w^1, w^2) = g(u) e_1 + f(u) \tilde{l}^I(w^1, w^2).$$

The second type rotational hypersurface $\widetilde{\mathcal{M}}^{II}$, obtained by the rotation of the meridian curve m about Oe_1 , is parametrized as follows:

$\widetilde{\mathcal{M}}^{II} : Z(u, w^1, w^2) = g(u) e_1 + f(u) (\sinh w^1 e_2 + \cosh w^1 \cos w^2 e_3 + \cosh w^1 \sin w^2 e_4)$,
or equivalently,

$$\widetilde{\mathcal{M}}^{II} : Z(u, w^1, w^2) = g(u) e_1 + f(u) \tilde{l}^{II}(w^1, w^2),$$

where $\tilde{l}^{II}(w^1, w^2) = \sinh w^1 e_2 + \cosh w^1 \cos w^2 e_3 + \cosh w^1 \sin w^2 e_4$ is the unit position vector of the hyperbolic sphere $\mathbb{H}_1^2(-1)$ in $\mathbb{E}_2^3 = \text{span}\{e_2, e_3, e_4\}$ centered at the origin O .

Now, we shall consider Lorentz surfaces in \mathbb{E}_2^4 which are one-parameter systems of meridians of the rotational hypersurface $\widetilde{\mathcal{M}}^I$ or $\widetilde{\mathcal{M}}^{II}$.

Meridian surfaces on $\widetilde{\mathcal{M}}^I$:

Let $c : l = l(v) = \tilde{l}^I(w^1(v), w^2(v))$, $v \in J$, $J \subset \mathbb{R}$ be a smooth curve on $\mathbb{S}_2^2(1)$. We consider the two-dimensional surface $\widetilde{\mathcal{M}}'$ lying on $\widetilde{\mathcal{M}}^I$ and defined by:

$$(13) \quad \widetilde{\mathcal{M}}' : z(u, v) = g(u) e_1 + f(u) l(v), \quad u \in I, v \in J.$$

The surface $\widetilde{\mathcal{M}}'$ is a one-parameter system of meridians of $\widetilde{\mathcal{M}}^I$. It can easily be seen that the surface \mathcal{M}'' , defined by (10), can be transformed into the surface $\widetilde{\mathcal{M}}'$ by the transformation T given by

$$(14) \quad T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

So, the meridian surfaces \mathcal{M}'' and $\widetilde{\mathcal{M}}'$ are congruent. Hence, all results concerning the surface \mathcal{M}'' hold true for the surface $\widetilde{\mathcal{M}}'$.

Meridian surfaces on $\widetilde{\mathcal{M}}^{II}$:

Now we shall consider meridian surfaces lying on the second type rotational hypersurface $\widetilde{\mathcal{M}}^{II}$.

Let $c : l = l(v) = \tilde{l}^{II}(w^1(v), w^2(v))$, $v \in J$, $J \subset \mathbb{R}$ be a smooth curve on the hyperbolic sphere $\mathbb{H}_1^2(-1)$ in \mathbb{E}_2^3 . We consider the meridian surface $\widetilde{\mathcal{M}}''$ lying on $\widetilde{\mathcal{M}}^{II}$ and defined by:

$$(15) \quad \widetilde{\mathcal{M}}'' : z(u, v) = g(u) e_1 + f(u) l(v), \quad u \in I, v \in J.$$

The tangent space of $\widetilde{\mathcal{M}}''$ is spanned by the vector fields

$$z_u = g'(u) e_1 + f'(u) l(v); \quad z_v = f(u) l'(v),$$

so, the coefficients of the first fundamental form are

$$E = g'^2 - f'^2; \quad F = 0; \quad G = f^2 \langle l', l' \rangle.$$

Now, we consider the following two cases:

Case (a): Let c be a spacelike curve, i.e. $\langle l', l' \rangle = 1$. In this case we suppose that $f'^2 - g'^2 = 1$. Then the coefficients of the first fundamental form are $E = -1$; $F = 0$; $G = f^2$. We shall denote the meridian surface in this case by $\widetilde{\mathcal{M}}''_a$. Under the transformation T given by (14) the surface \mathcal{M}'_b is transformed into the surface $\widetilde{\mathcal{M}}''_a$.

Case (b): Let the curve c be timelike, i.e. $\langle l', l' \rangle = -1$. In this case we assume that $f'^2 - g'^2 = -1$. Then for the coefficients of the first fundamental form we have $E = 1$; $F = 0$; $G = -f^2$. We denote the meridian surface in this case by $\widetilde{\mathcal{M}}''_b$. It is clear that the meridian surfaces \mathcal{M}'_a and $\widetilde{\mathcal{M}}''_b$ are congruent (up to the transformation T).

In the present paper we will study three types of Lorentz meridian surfaces in \mathbb{E}_2^4 , namely the surfaces denoted by \mathcal{M}'_a , \mathcal{M}'_b , and \mathcal{M}'' .

4. CLASSIFICATION OF MERIDIAN SURFACES WITH PARALLEL MEAN CURVATURE VECTOR FIELD

In this section we shall describe all meridian surfaces defined in the previous section which have parallel mean curvature vector field.

First we consider the meridian surface \mathcal{M}'_a , defined by (1), where $f'^2 - g'^2 = -1$. Using formulas (2) and (3), we get

$$(16) \quad \begin{aligned} \widetilde{\nabla}_X n_1 &= 0; & \widetilde{\nabla}_X n_2 &= \kappa_m X; \\ \widetilde{\nabla}_Y n_1 &= -\frac{\kappa}{f} Y; & \widetilde{\nabla}_Y n_2 &= \frac{g'}{f} Y. \end{aligned}$$

The mean curvature vector field H of the meridian surface \mathcal{M}'_a is given by formula (6). Hence, by use of (16) we obtain

$$(17) \quad \begin{aligned} \widetilde{\nabla}_X H &= -\frac{f''(ff''+f'^2+1)}{2f(f'^2+1)} X + \frac{\kappa f'}{2f^2} n_1 - \left(\frac{ff''+f'^2+1}{2fg'}\right)' n_2; \\ \widetilde{\nabla}_Y H &= \frac{\kappa^2 - (ff''+f'^2+1)}{2f^2} Y - \frac{\kappa'}{2f^2} n_1. \end{aligned}$$

Theorem 4.1. *Let \mathcal{M}'_a be a meridian surface on \mathcal{M}^I defined by (1), (resp. $\widetilde{\mathcal{M}}''_b$ be a meridian surface on $\widetilde{\mathcal{M}}^{II}$ defined by (15)), where $f'^2 - g'^2 = -1$. Then \mathcal{M}'_a (resp. $\widetilde{\mathcal{M}}''_b$) has parallel mean curvature vector field if and only if one of the following cases holds:*

(i) *the curve c has constant spherical curvature and the meridian m is defined by $f(u) = a$, $g(u) = \pm u + b$, where $a = \text{const} \neq 0$, $b = \text{const}$. In this case \mathcal{M}'_a (resp. $\widetilde{\mathcal{M}}''_b$) is a flat CMC-surface lying in a hyperplane of \mathbb{E}_2^4 .*

(ii) *the curve c has zero spherical curvature and the meridian m is determined by $f' = \varphi(f)$ where*

$$\varphi(t) = \pm \frac{1}{t} \sqrt{(c \pm at^2)^2 - t^2}, \quad a = \text{const} \neq 0, \quad c = \text{const},$$

$g(u)$ is defined by $g' = \sqrt{f'^2 + 1}$. In this case \mathcal{M}'_a (resp. $\widetilde{\mathcal{M}}''_b$) is a non-flat CMC-surface lying in a hyperplane of \mathbb{E}_2^4 .

Proof. Let \mathcal{M}'_a be a surface with parallel mean curvature vector field. Using formulas (17) we get the following conditions

$$(18) \quad \begin{aligned} \kappa' &= 0; \\ \kappa f' &= 0; \\ \left(\frac{ff''+f'^2+1}{2fg'}\right)' &= 0. \end{aligned}$$

The first equality of (18) implies that the spherical curvature κ of c is constant. Having in mind (18) we obtain that there are two possible cases:

Case (i): $f' = 0$, i.e. $f(u) = a$, $a = \text{const} \neq 0$. Using that $f'^2 - g'^2 = -1$, we get $g(u) = \pm u + b$, $b = \text{const}$. In this case the mean curvature vector field is expressed as follows

$$H = -\frac{\kappa}{2a} n_1 \mp \frac{1}{2a} n_2.$$

The last equality implies that $\langle H, H \rangle = \frac{1-\kappa^2}{4a^2} = \text{const}$. Hence, \mathcal{M}'_a has constant mean curvature. If $\kappa^2 = 1$, then \mathcal{M}'_a is quasi-minimal. If $\kappa^2 \neq 1$, then \mathcal{M}'_a has non-zero constant mean curvature. Having in mind that the Gauss curvature of \mathcal{M}'_a is expressed by formula (5), in this case we obtain $K = 0$, i.e. \mathcal{M}'_a is flat. Up to reparametrization we can assume that $g(u) = u$. Then, since $\kappa_m = 0$, from (16) we get

$$(19) \quad \begin{aligned} \widetilde{\nabla}_X n_1 &= 0; & \widetilde{\nabla}_X n_2 &= 0; \\ \widetilde{\nabla}_Y n_1 &= -\frac{\kappa}{a} Y; & \widetilde{\nabla}_Y n_2 &= \frac{1}{a} Y. \end{aligned}$$

If $\kappa = 0$, then $\tilde{\nabla}_X n_1 = \tilde{\nabla}_Y n_1 = 0$, which implies that the normal vector field n_1 is constant. Hence, \mathcal{M}'_a lies in a constant 3-dimensional space \mathbb{E}_1^3 parallel to $\text{span}\{X, Y, n_2\}$. If $\kappa \neq 0$ and $\kappa^2 < 1$, we consider the normal vector fields $N_1 = \frac{1}{\sqrt{1-\kappa^2}}(n_1 + \kappa n_2)$; $N_2 = \frac{1}{\sqrt{1-\kappa^2}}(\kappa n_1 + n_2)$. Using (19) we obtain $\tilde{\nabla}_X N_1 = \tilde{\nabla}_Y N_1 = 0$, and hence, the surface \mathcal{M}'_a lies in a hyperplane \mathbb{E}_1^3 parallel to $\text{span}\{X, Y, N_2\}$. If $\kappa^2 > 1$, we consider the normal vector fields $N_1 = \frac{1}{\sqrt{\kappa^2-1}}(\kappa n_1 + n_2)$; $N_2 = \frac{1}{\sqrt{\kappa^2-1}}(n_1 + \kappa n_2)$ and obtain $\tilde{\nabla}_X N_2 = \tilde{\nabla}_Y N_2 = 0$, which imply that \mathcal{M}'_a lies in a hyperplane \mathbb{E}_2^3 parallel to $\text{span}\{X, Y, N_1\}$. If $\kappa^2 = 1$, we consider $N_1 = \frac{n_1+n_2}{\sqrt{2}}$; $N_2 = \frac{n_1-n_2}{\sqrt{2}}$ and using (19) we get

$$\begin{aligned}\tilde{\nabla}_X N_1 &= 0; & \tilde{\nabla}_X N_2 &= 0; \\ \tilde{\nabla}_Y N_1 &= \frac{1-\kappa}{\sqrt{2a}} Y; & \tilde{\nabla}_Y N_2 &= -\frac{1+\kappa}{\sqrt{2a}} Y.\end{aligned}$$

In the case $\kappa = 1$ we get $N_1 = \text{const}$ and hence \mathcal{M}'_a lies in a degenerate hyperplane parallel to $\text{span}\{X, Y, N_2\}$. In the case $\kappa = -1$ we get $N_2 = \text{const}$ and hence \mathcal{M}'_a lies in a degenerate hyperplane parallel to $\text{span}\{X, Y, N_1\}$. Finally, we obtain that in all cases for κ the surface \mathcal{M}'_a is a flat CMC-surface lying in a hyperplane of \mathbb{E}_2^4 .

Case (ii): $\kappa = 0$ and $\frac{ff''+f'^2+1}{2fg'} = a = \text{const}$. If $a = 0$, then $H = 0$, i.e. \mathcal{M}'_a is minimal. Since we consider non-minimal surfaces, we assume that $a \neq 0$. In this case the meridian m is determined by the following differential equation:

$$(20) \quad ff'' + f'^2 + 1 = \pm 2af\sqrt{f'^2 + 1}, \quad a = \text{const} \neq 0.$$

The solutions of the above differential equation can be found in the following way. Setting $f' = \varphi(f)$ in equation (20), we obtain that the function $\varphi = \varphi(t)$ is a solution of the equation:

$$(21) \quad \frac{t}{2}(\varphi^2)' + \varphi^2 + 1 = \pm 2at\sqrt{\varphi^2 + 1}.$$

If we set $z(t) = \sqrt{\varphi^2(t) + 1}$, equation (21) takes the form

$$z' + \frac{1}{t}z = \pm 2a.$$

The general solution of the last equation is given by the formula $z(t) = \frac{c \pm at^2}{t}$, $c = \text{const}$. Hence, the general solution of (21) is

$$\varphi(t) = \pm \frac{1}{t} \sqrt{(c \pm at^2)^2 - t^2}.$$

Since $\kappa = 0$, from (16) it follows that $\tilde{\nabla}_X n_1 = \tilde{\nabla}_Y n_1 = 0$. Hence, \mathcal{M}'_a lies in a 3-dimensional constant hyperplane parallel to $\text{span}\{X, Y, n_2\}$. Moreover, $K \neq 0$ and $\langle H, H \rangle = a^2 = \text{const}$, so \mathcal{M}'_a is a non-flat CMC-surface lying in a hyperplane \mathbb{E}_1^3 of \mathbb{E}_2^4 .

Conversely, if one of the cases (i) or (ii) stated in the theorem holds true, then by direct computation we get that $D_X H = D_Y H = 0$, i.e. the surface has parallel mean curvature vector field. \square

Next, we consider the meridian surface \mathcal{M}'_b , defined by (1), where $f'^2 - g'^2 = 1$. The mean curvature vector field of \mathcal{M}'_b is given by formula (9). Similarly to the considerations about the meridian surface \mathcal{M}'_a , now we obtain

$$(22) \quad \begin{aligned} \tilde{\nabla}_X H &= \frac{f''(ff''+f'^2-1)}{2f(f'^2-1)} X + \frac{\kappa f'}{2f^2} n_1 + \left(\frac{ff''+f'^2-1}{2fg'}\right)' n_2; \\ \tilde{\nabla}_Y H &= \frac{ff''+f'^2-1-\kappa^2}{2f^2} Y - \frac{\kappa'}{2f^2} n_1. \end{aligned}$$

In the following theorem we give the classification of the meridian surfaces of type \mathcal{M}'_b having parallel mean curvature vector field.

Theorem 4.2. *Let \mathcal{M}'_b be a meridian surface on \mathcal{M}^I defined by (1) (resp. $\tilde{\mathcal{M}}''_a$ be a meridian surface on $\tilde{\mathcal{M}}^{II}$ defined by (15)), where $f'^2 - g'^2 = 1$. Then \mathcal{M}'_b (resp. $\tilde{\mathcal{M}}''_a$) has parallel mean curvature vector field if and only if the curve c has zero spherical curvature and the meridian m is determined by $f' = \varphi(f)$ where*

$$\varphi(t) = \pm \frac{1}{t} \sqrt{(c \pm at^2)^2 + t^2}, \quad a = \text{const} \neq 0, \quad c = \text{const},$$

$g(u)$ is defined by $g' = \sqrt{f'^2 - 1}$. Moreover, \mathcal{M}'_b (resp. $\tilde{\mathcal{M}}''_a$) is a non-flat CMC-surface lying in a hyperplane of \mathbb{E}_2^4 .

Proof. Let \mathcal{M}'_b be a surface with parallel mean curvature vector field. Formulas (22) imply the following conditions

$$\begin{aligned} \kappa' &= 0; \\ \kappa f' &= 0; \\ \left(\frac{ff''+f'^2-1}{2fg'}\right)' &= 0, \end{aligned}$$

and hence, we get $\kappa = \text{const}$. If we assume that $f' = 0$, i.e. $f(u) = a = \text{const}$, then having in mind that $f'^2 - g'^2 = 1$, we get $g'^2 = -1$, which is not possible. So, the only possible case is $\kappa = 0$ and $\frac{ff''+f'^2-1}{2fg'} = a = \text{const}$. We consider non-minimal surfaces, so we assume that $a \neq 0$. The meridian m is determined by the following differential equation:

$$(23) \quad ff'' + f'^2 - 1 = \pm 2af\sqrt{f'^2 - 1}, \quad a = \text{const} \neq 0.$$

Similarly to the proof of Theorem 4.1, setting $f' = \varphi(f)$ in equation (23), we obtain

$$(24) \quad \varphi(t) = \pm \frac{1}{t} \sqrt{(c \pm at^2)^2 + t^2}, \quad a = \text{const} \neq 0, \quad c = \text{const}.$$

Since $\kappa = 0$, we have $\tilde{\nabla}_X n_1 = \tilde{\nabla}_Y n_1 = 0$. So, \mathcal{M}'_b lies in a 3-dimensional constant hyperplane parallel to $\text{span}\{X, Y, n_2\}$. Moreover, $K \neq 0$ and $\langle H, H \rangle = \text{const}$. Hence, \mathcal{M}'_b is a non-flat CMC-surface.

Conversely, if $\kappa = 0$ and the meridian m is determined by (24), then direct computation show that $D_X H = D_Y H = 0$, i.e. \mathcal{M}'_b has parallel mean curvature vector field. \square

Now, let us consider the meridian surface \mathcal{M}'' , defined by (10), where $f'^2 + g'^2 = 1$. The mean curvature vector field of \mathcal{M}'' is given by formula (12). The derivatives of H with respect to X and Y are given by the following formulas

$$\begin{aligned}\tilde{\nabla}_X H &= \frac{f''(ff''+f'^2-1)}{2f(1-f'^2)} X - \frac{\kappa f'}{2f^2} n_1 + \left(\frac{ff''+f'^2-1}{2fg'}\right)' n_2; \\ \tilde{\nabla}_Y H &= -\frac{ff''+f'^2-1+\kappa^2}{2f^2} Y + \frac{\kappa'}{2f^2} n_1.\end{aligned}$$

Similarly to the proof of Theorem 4.1, we obtain the following classification result.

Theorem 4.3. *Let \mathcal{M}'' be a meridian surface on \mathcal{M}^{II} defined by (10) (resp. $\tilde{\mathcal{M}}'$ be a meridian surface on $\tilde{\mathcal{M}}^I$, defined by (13)), where $f'^2 + g'^2 = 1$. Then \mathcal{M}'' (resp. $\tilde{\mathcal{M}}'$) has parallel mean curvature vector field if and only if one of the following cases holds:*

(i) *the curve c has constant spherical curvature and the meridian m is defined by $f(u) = a$, $g(u) = \pm u + b$, where $a = \text{const} \neq 0$, $b = \text{const}$. In this case \mathcal{M}'' (resp. $\tilde{\mathcal{M}}'$) is a flat CMC-surface lying in a hyperplane of \mathbb{E}_2^4 .*

(ii) *the curve c has zero spherical curvature and the meridian m is determined by $f' = \varphi(f)$ where*

$$\varphi(t) = \pm \frac{1}{t} \sqrt{t^2 - (c \pm at^2)^2}, \quad a = \text{const} \neq 0, \quad c = \text{const},$$

$g(u)$ is defined by $g' = \sqrt{1 - f'^2}$. In this case \mathcal{M}'' (resp. $\tilde{\mathcal{M}}'$) is a non-flat CMC-surface lying in a hyperplane of \mathbb{E}_2^4 .

Theorems 4.1, 4.2, and 4.3 show that each meridian surface with parallel mean curvature vector field is a CMC-surface and lies in a hyperplane of \mathbb{E}_2^4 . So, we have the following result.

Corollary 4.1. *There are no Lorentz meridian surfaces with parallel mean curvature vector field other than CMC-surfaces lying in a hyperplane of \mathbb{E}_2^4 .*

5. CLASSIFICATION OF MERIDIAN SURFACES WITH PARALLEL NORMALIZED MEAN CURVATURE VECTOR FIELD

In this section we give the classification of all meridian surfaces which have parallel normalized mean curvature vector field but not parallel H .

First we consider the meridian surface \mathcal{M}'_a , defined by (1), where $f'^2 - g'^2 = -1$. The mean curvature vector field H of \mathcal{M}'_a is given by formula (6). We assume that $\langle H, H \rangle \neq 0$, i.e. $(ff'' + f'^2 + 1)^2 - \kappa^2(f'^2 + 1) \neq 0$, and denote $\varepsilon = \text{sign}\langle H, H \rangle$. Then the normalized mean curvature vector field of \mathcal{M}'_a is given by

$$(25) \quad H_0 = \frac{1}{\sqrt{\varepsilon((ff'' + f'^2 + 1)^2 - \kappa^2(f'^2 + 1))}} \left(-\kappa\sqrt{f'^2 + 1} n_1 - (ff'' + f'^2 + 1) n_2 \right).$$

If $\kappa = 0$, then $H_0 = n_2$ and (16) implies that $D_X H_0 = D_Y H_0 = 0$, i.e. H_0 is parallel in the normal bundle. We consider this case as trivial, since under the assumption $\kappa = 0$ the surface \mathcal{M}'_a lies in a 3-dimensional space

\mathbb{E}_1^3 and every surface in \mathbb{E}_1^3 has parallel normalized mean curvature vector field. So, further we assume that $\kappa \neq 0$.

For simplicity we denote

$$A = \frac{-\kappa\sqrt{f'^2+1}}{\sqrt{\varepsilon((ff''+f'^2+1)^2-\kappa^2(f'^2+1))}}, \quad B = \frac{-(ff''+f'^2+1)}{\sqrt{\varepsilon((ff''+f'^2+1)^2-\kappa^2(f'^2+1))}},$$

so, the normalized mean curvature vector is expressed as $H_0 = An_1 + Bn_2$. Then equalities (25) and (16) imply

$$(26) \quad \begin{aligned} \tilde{\nabla}_X H_0 &= B\kappa_m X + A'_u n_1 + B'_u n_2; \\ \tilde{\nabla}_Y H_0 &= \frac{Bg'-A\kappa}{f} Y + \frac{A'_v}{f} n_1 + \frac{B'_v}{f} n_2, \end{aligned}$$

where A'_u (resp. A'_v) denotes $\frac{\partial A}{\partial u}$ (resp. $\frac{\partial A}{\partial v}$).

Theorem 5.1. *Let \mathcal{M}'_a be a meridian surface on \mathcal{M}^I defined by (1), (resp. $\widetilde{\mathcal{M}}''_b$ be a meridian surface on $\widetilde{\mathcal{M}}^{II}$ defined by (15)), where $f'^2 - g'^2 = -1$. Then \mathcal{M}'_a (resp. $\widetilde{\mathcal{M}}''_b$) has parallel normalized mean curvature vector field but not parallel mean curvature vector if and only if one of the following cases holds:*

(i) $\kappa \neq 0$ and the meridian m is defined by

$$f(u) = \pm\sqrt{-u^2 + 2au + b}, \quad g(u) = \pm\sqrt{a^2 + b} \arcsin \frac{u-a}{\sqrt{a^2 + b}} + c,$$

where $a = \text{const}$, $b = \text{const}$, $c = \text{const}$.

(ii) the curve c has non-zero constant spherical curvature and the meridian m is determined by $f' = \varphi(f)$ where

$$\varphi(t) = \pm \frac{1}{t} \sqrt{(ct+a)^2 - t^2}, \quad a = \text{const}, \quad c = \text{const} \neq 0, \quad c^2 \neq \kappa^2,$$

$g(u)$ is defined by $g' = \sqrt{f'^2 + 1}$.

Proof. Let \mathcal{M}'_a be a surface with parallel normalized mean curvature vector field, i.e. $D_X H_0 = D_Y H_0 = 0$. Then from (26) it follows that $A = \text{const}$, $B = \text{const}$. Hence,

$$(27) \quad \begin{aligned} \frac{-\kappa\sqrt{f'^2+1}}{\sqrt{\varepsilon((ff''+f'^2+1)^2-\kappa^2(f'^2+1))}} &= \alpha = \text{const}; \\ \frac{-(ff''+f'^2+1)}{\sqrt{\varepsilon((ff''+f'^2+1)^2-\kappa^2(f'^2+1))}} &= \beta = \text{const}. \end{aligned}$$

Case (i): $ff'' + f'^2 + 1 = 0$. In this case, from (25) we get that the normalized mean curvature vector field is $H_0 = n_1$ and the mean curvature vector field is $H = -\frac{\kappa}{2f} n_1$. Since we study surfaces with $\langle H, H \rangle \neq 0$, we get $\kappa \neq 0$. The solution of the differential equation $ff'' + f'^2 + 1 = 0$ is given by the formula $f(u) = \pm\sqrt{-u^2 + 2au + b}$, where $a = \text{const}$, $b = \text{const}$. Using that $g' = \sqrt{f'^2 + 1}$, we obtain the following equation for $g(u)$:

$$g' = \pm \frac{\sqrt{a^2 + b}}{\sqrt{-u^2 + 2au + b}}.$$

Integrating the above equation we get

$$g(u) = \pm \sqrt{a^2 + b} \arcsin \frac{u - a}{\sqrt{a^2 + b}} + c, \quad c = \text{const.}$$

Case (ii): $ff'' + f'^2 + 1 \neq 0$. From (27) we get

$$(28) \quad \frac{\beta}{\alpha} \kappa = \frac{ff'' + f'^2 + 1}{\sqrt{f'^2 + 1}}, \quad \alpha \neq 0, \beta \neq 0.$$

Since the left-hand side of equality (28) is a function of v , the right-hand side of (28) is a function of u , we obtain that

$$\begin{aligned} \frac{ff'' + f'^2 + 1}{\sqrt{f'^2 + 1}} &= c, \quad c = \text{const} \neq 0; \\ \kappa &= \frac{\alpha}{\beta} c = \text{const.} \end{aligned}$$

Then the length of the mean curvature vector field is $\langle H, H \rangle = \frac{c^2 - \kappa^2}{4f^2}$. Since we study surfaces with $\langle H, H \rangle \neq 0$, we get $c^2 \neq \kappa^2$. The meridian m is determined by the following differential equation:

$$(29) \quad ff'' + f'^2 + 1 = c\sqrt{f'^2 + 1}.$$

The solutions of this differential equation can be found as follows. We set $f' = \varphi(t)$ in equation (29) and obtain that the function $\varphi = \varphi(t)$ satisfies

$$(30) \quad \frac{t}{2} (\varphi^2)' + \varphi^2 + 1 = c\sqrt{\varphi^2 + 1}.$$

Putting $z(t) = \sqrt{\varphi^2(t) + 1}$, equation (30) can be written as

$$z' + \frac{1}{t} z = \frac{c}{t},$$

whose general solution is $z(t) = \frac{ct+a}{t}$, $a = \text{const}$. Hence, the general solution of (30) is given by the formula

$$\varphi(t) = \pm \frac{1}{t} \sqrt{(ct + a)^2 - t^2}.$$

Conversely, if one of the cases (i) or (ii) stated in the theorem holds true, then by direct computation we get that $D_X H_0 = D_Y H_0 = 0$, i.e. the surface has parallel normalized mean curvature vector field. Moreover, in case (i) we have

$$D_X H = \frac{\kappa f'}{2f^2} n_1; \quad D_Y H = -\frac{\kappa'}{2f^2} n_1,$$

which implies that H is not parallel in the normal bundle, since $\kappa \neq 0$, $f' \neq 0$. In case (ii) we get

$$D_X H = \frac{\kappa f'}{2f^2} n_1 + \frac{c f'}{2f^2} n_2; \quad D_Y H = 0,$$

and again we have that H is not parallel in the normal bundle.

In a similar way we consider the meridian surface \mathcal{M}'_b , defined by (1), where $f'^2 - g'^2 = 1$. The normalized mean curvature vector field of \mathcal{M}'_b is given by

$$H_0 = \frac{1}{\sqrt{\varepsilon(\kappa^2(f'^2 - 1) - (ff'' + f'^2 + 1)^2)}} \left(-\kappa\sqrt{f'^2 - 1} n_1 + (ff'' + f'^2 - 1) n_2 \right),$$

where $\varepsilon = \text{sign}\langle H, H \rangle$ and we assume that $\kappa^2(f'^2 - 1) - (ff'' + f'^2 - 1)^2 \neq 0$.

The classification of the meridian surfaces of type \mathcal{M}'_b and $\widetilde{\mathcal{M}}''_a$ which have parallel normalized mean curvature vector field but not parallel H is given in the following theorem.

Theorem 5.2. *Let \mathcal{M}'_b be a meridian surface on \mathcal{M}^I defined by (1) (resp. $\widetilde{\mathcal{M}}''_a$ be a meridian surface on $\widetilde{\mathcal{M}}^{II}$ defined by (15)), where $f'^2 - g'^2 = 1$. Then \mathcal{M}'_b (resp. $\widetilde{\mathcal{M}}''_a$) has parallel normalized mean curvature vector field but not parallel mean curvature vector if and only if one of the following cases holds:*

(i) $\kappa \neq 0$ and the meridian m is defined by

$$f(u) = \pm\sqrt{u^2 + 2au + b}, \quad g(u) = \pm\sqrt{a^2 - b} \ln |u + a + \sqrt{u^2 + 2au + b}| + c,$$

where $a = \text{const}$, $b = \text{const}$, $c = \text{const}$, $a^2 - b > 0$.

(ii) the curve c has non-zero constant spherical curvature and the meridian m is determined by $f' = \varphi(f)$ where

$$\varphi(t) = \pm\frac{1}{t}\sqrt{(ct + a)^2 + t^2}, \quad a = \text{const}, \quad c = \text{const} \neq 0, \quad c^2 \neq \kappa^2,$$

$g(u)$ is defined by $g' = \sqrt{f'^2 - 1}$.

The proof of this theorem is similar to the proof of Theorem 5.1.

The classification of the meridian surfaces of type \mathcal{M}'' and $\widetilde{\mathcal{M}}'$ which have parallel normalized mean curvature vector field but not parallel H is given in the following theorem.

Theorem 5.3. *Let \mathcal{M}'' be a meridian surface on \mathcal{M}^{II} defined by (10) (resp. $\widetilde{\mathcal{M}}'$ be a meridian surface on $\widetilde{\mathcal{M}}^I$, defined by (13)), where $f'^2 + g'^2 = 1$. Then \mathcal{M}'' (resp. $\widetilde{\mathcal{M}}'$) has parallel normalized mean curvature vector field but not parallel mean curvature vector if and only if one of the following cases holds:*

(i) $\kappa \neq 0$ and the meridian m is defined by

$$f(u) = \pm\sqrt{u^2 + 2au + b}, \quad g(u) = \pm\sqrt{b - a^2} \ln |u + a + \sqrt{u^2 + 2au + b}| + c,$$

where $a = \text{const}$, $b = \text{const}$, $c = \text{const}$, $b - a^2 > 0$.

(ii) the curve c has non-zero constant spherical curvature and the meridian m is determined by $f' = \varphi(f)$ where

$$\varphi(t) = \pm\frac{1}{t}\sqrt{t^2 - (a - ct)^2}, \quad a = \text{const}, \quad c = \text{const} \neq 0, \quad c^2 \neq \kappa^2,$$

$g(u)$ is defined by $g' = \sqrt{1 - f'^2}$.

Remark 5.1. *All theorems stated in this section give examples of Lorentz surfaces in the pseudo-Euclidean space \mathbb{E}_2^4 which have parallel normalized mean curvature vector field but not parallel mean curvature vector field.*

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REFERENCES

- [1] Arslan K., Bulca B, and Milousheva V., *Meridian surfaces in \mathbb{E}^4 with pointwise 1-type Gauss map*, Bull. Korean Math. Soc., **51(3)(2014)**, 911–922.
- [2] Arslan K. and Milousheva V., *Meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map in Minkowski 4-space*, Taiwanese J. Math., **20(2)(2016)**, 311–332.
- [3] Bulca B. and Milousheva V., *Meridian surfaces with constant mean curvature in pseudo-Euclidean 4-space with neutral metric*, Mediterr. J. Math., **14(2)(2017)**, DOI:10.1007/s00009-017-0878-x (Published online: 02 March 2017).
- [4] Chen B.-Y., *Geometry of Submanifolds*, Marcel Dekker, Inc., New York, 1973.
- [5] Chen B.-Y., *Surfaces with parallel normalized mean curvature vector*, Monatsh. Math., **90(3)(1980)**, 185–194.
- [6] Chen B.-Y., *Classification of spatial surfaces with parallel mean curvature vector in pseudo-Euclidean spaces with arbitrary codimension*, J. Math. Phys., **50(4)(2009)**, 043503.
- [7] Chen B.-Y., *Complete classification of spatial surfaces with parallel mean curvature vector in arbitrary non-flat pseudo-Riemannian space forms*, Cent. Eur. J. Math., **7(2009)**, 400–428.
- [8] Chen B.-Y., *Complete classification of Lorentz surfaces with parallel mean curvature vector in arbitrary pseudo-Euclidean space*, Kyushu J. Math., **64(2)(2010)**, 261–279.
- [9] Chen B.-Y., *Submanifolds with parallel mean curvature vector in Riemannian and indefinite space forms*, Arab J. Math. Sci., **16(1)(2010)**, 1–46.
- [10] Chen B.-Y., *Pseudo-Riemannian Geometry, δ -invariants and Applications*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
- [11] Fu Y. and Hou Z.-H., *Classification of Lorentzian surfaces with parallel mean curvature vector in pseudo-Euclidean spaces*, J. Math. Anal. Appl., **371(1)(2010)**, 25–40.
- [12] Ganchev G. and Milousheva V., *Invariants and Bonnet-type Theorem for Surfaces in \mathbb{R}^4* , Cent. Eur. J. Math., **8(2010)**, 993–1008.
- [13] Ganchev G. and Milousheva V., *An Invariant Theory of Marginally Trapped Surfaces in the Four-dimensional Minkowski Space*, J. Math. Phys., **53(2012)**, 033705.
- [14] Ganchev G. and Milousheva V., *Special Classes of Meridian Surfaces in the Four-dimensional Euclidean Space*, Bull. Korean Math. Soc., **52(6)(2015)**, 2035–2045.
- [15] Ganchev G. and Milousheva V., *Meridian Surfaces of Elliptic or Hyperbolic Type in the Four-dimensional Minkowski Space*, Math. Commun., **21(1)(2016)**, 1–21.
- [16] O'Neill B., *Semi-Riemannian geometry with applications to relativity*, Academic Press, London, 1983.
- [17] Rosca, R., *On null hypersurfaces of a Lorentzian manifold*, Tensor(N.S.), **23(1972)**, 66–74.
- [18] Yau, S., *Submanifolds with constant mean curvature*, Amer. J. Math., **96(1974)**, 346–366.

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