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INEQUALITIES IN QUADRILATERAL INVOLVING THE NEWTON LINE

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Abstract. We derive new inequalities from a recently strengthened inequality for the perimeter of a quadrilateral.

1. Introduction

It is well known that the perimeter of a convex quadrilateral is greater than the sum of the diagonals, which follows easily from repeated application of the triangle inequality. In [4] the following problem was proposed:

"*Let $ABCD$ be a convex quadrilateral. Let E be the midpoint of AC, and let F be the midpoint of BD. Show that $|AB| + |BC| + |CD| + |DA| \geq |AC| + |BD| + 2|EF|$. (Here $|XY|$ denotes the distance from X to Y.)*"

In words, the perimeter of a convex quadrilateral is at least equal to the sum of the diagonals plus twice the length of the so-called Newton line. This statement is quite remarkable due to the inclusion of the last term. The length of the Newton line is a measure of how much a quadrilateral deviates from a parallelogram [1]. Quadrilaterals can be constructed with an arbitrarily large Newton line [2], making this result all the more powerful.

While the published solution has yet to appear, an online solution [7] reveals that the inequality follows from a direct application of Hlawka's inequality and in fact holds for any quadrilateral.

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At the same time, the criteria for equality in Hlawka's inequality [6], while admittedly complicated, demonstrate that the above inequality must be strict or else the convex quadrilateral degenerates into a triangle (at best). We will use this new inequality, in its strict form, to derive other, apparently new inequalities.

2. Convex quadrilateral

Let $|AB| = a, |BC| = b, |CD| = c, |DA| = d, |AC| = p, |BD| = q, |EF| = v$. So for the rest of this paper, a, b, c, d are consecutive sides of the quadrilateral, p and q are the diameters as defined, and v is the length of the line segment (the Newton line) connecting the midpoints of the diagonals. The above inequality, due to L. Giugiuc, can now be restated, in its strict form, as follows.

Theorem 2.1. *For a convex quadrilateral,*

$$a + b + c + d > p + q + 2v. \tag{1}$$

Because the perimeter of the Varignon parallelogram (formed by connecting the midpoints of consecutive sides) of a convex quadrilateral with diagonals p and q is equal to $p + q$, we can say the following.

Corollary 2.1. *The perimeter of a convex quadrilateral exceeds that of its Varignon parallelogram by more than $2v$.*

Theorem 2.2. *For a convex quadrilateral,*

$$ab + ac + ad + bc + bd + cd > pq + 2v(p + q). \tag{2}$$

Proof. Square both sides of (1), then remove $a^2 + b^2 + c^2 + d^2$ from the left side and $p^2+q^2+4v^2$ from the right side since they are equal by Euler's quadrilateral theorem. □

This result can be compared to the known inequality $ab+ac+ad+bc+bd+cd \geq 6K$, where K is the area of the quadrilateral, valid only for a bicentric quadrilateral (see, e.g., Wikipedia for this and all "known" results referred to in this paper).

3. Cyclic quadrilateral

In this section we consider convex quadrilaterals that are cyclic, that is, those with a circumcircle.

The following proposition, aside from being interesting in its own right, allows us to eliminate v from the inequalities to be derived, which may be useful when v is unavailable.

Proposition 3.1. *For a convex quadrilateral that is cyclic, $2v \geq |p - q|$.*

Proof. Let $v > 0$ and proceed as in the geometric proof of Euler's quadrilateral theorem (see [3]). Construct CG parallel and equal in length to AB, and CH parallel and equal in length to AD as shown in Figure 1. Note that $ABCG$, $ADCH$ and $BDGH$ are parallelograms, and because EF is the midline of triangle BDG, $|DG| = 2|EF|$. By the triangle inequality, $2v > c - a$ and $2v > b - d$. Squaring both sides of each of these inequalities and adding the inequalities together, $8v^2 > a^2 + b^2 + c^2 + d^2 - 2(ac + bd)$. Substitute $p^2 + q^2 + 4v^2$ for $a^2 + b^2 + c^2 + d^2$ by Euler's quadrilateral theorem, and pq for $ac + bd$ by Ptolemy's theorem for cyclic quadrilaterals. Simplifying, $4v^2 > (p - q)^2$. Taking the square root of both sides, the strict inequality follows. Obviously, equality is attained only when $v = 0$, which for a cyclic quadrilateral means only for a rectangle. ☐

Remark 3.1. *If $p = q$, the cyclic quadrilateral must be an isosceles trapezoid (see* [5, Theorem 17]*), the rectangle being a limiting case.*

Theorem 3.1. *For a convex quadrilateral that is cyclic,*

$$(a + c)(b + d) > 2v(p + q) \geq |p^2 - q^2|. \tag{3}$$

Proof. In (2), remove $ac + bd$ from the left side and pq from the right side since they are equal by Ptolemy's theorem. That gives the left inequality. Applying Proposition 3.1 gives the right inequality. ☐

This result can be compared to the known inequality $(a + c)(b + d) \geq 4K$, where K is the area of the quadrilateral, valid for any convex quadrilateral.

Theorem 3.2. *For a convex quadrilateral that is cyclic,*

$$ab + cd > 2vq \geq |p - q|q \text{ and } ad + bc > 2vp \geq |p - q|p. \tag{4}$$

Proof. Substituting the known cyclic identity $\dfrac{p}{q} = \dfrac{ad + bc}{ab + cd}$ (Ptolemy's second theorem) into $ad + bc$ of the expanded left side $ad + bc + ab + cd$ of (3), we get

$$\begin{aligned} \frac{p}{q}(ab + cd) + (ab + cd) &> 2v(p + q) \\ (ab + cd)\frac{p + q}{q} &> 2v(p + q) \\ (ab + cd) &> 2vq. \end{aligned}$$

The left inequality in the second statement of the theorem follows similarly by substituting the same cyclic identity into $ab + cd$ of the expanded left side of (3). Applying Proposition 3.1 gives the right inequalities. ☐

Corollary 3.1. *For K the area of a convex quadrilateral that is cyclic, A the angle formed by sides a and d, and B the angle formed by sides a and b,*

$$K > \max\{vp \sin A, vq \sin B\} \geq \frac{1}{2} \max\{|p - q|p \sin A, |p - q|q \sin B\}.$$

Proof. Insert the inequalities in (4) into the known cyclic identities $K = \frac{1}{2}(ad + bc) \sin A$ and $K = \frac{1}{2}(ab + cd) \sin B$. ☐

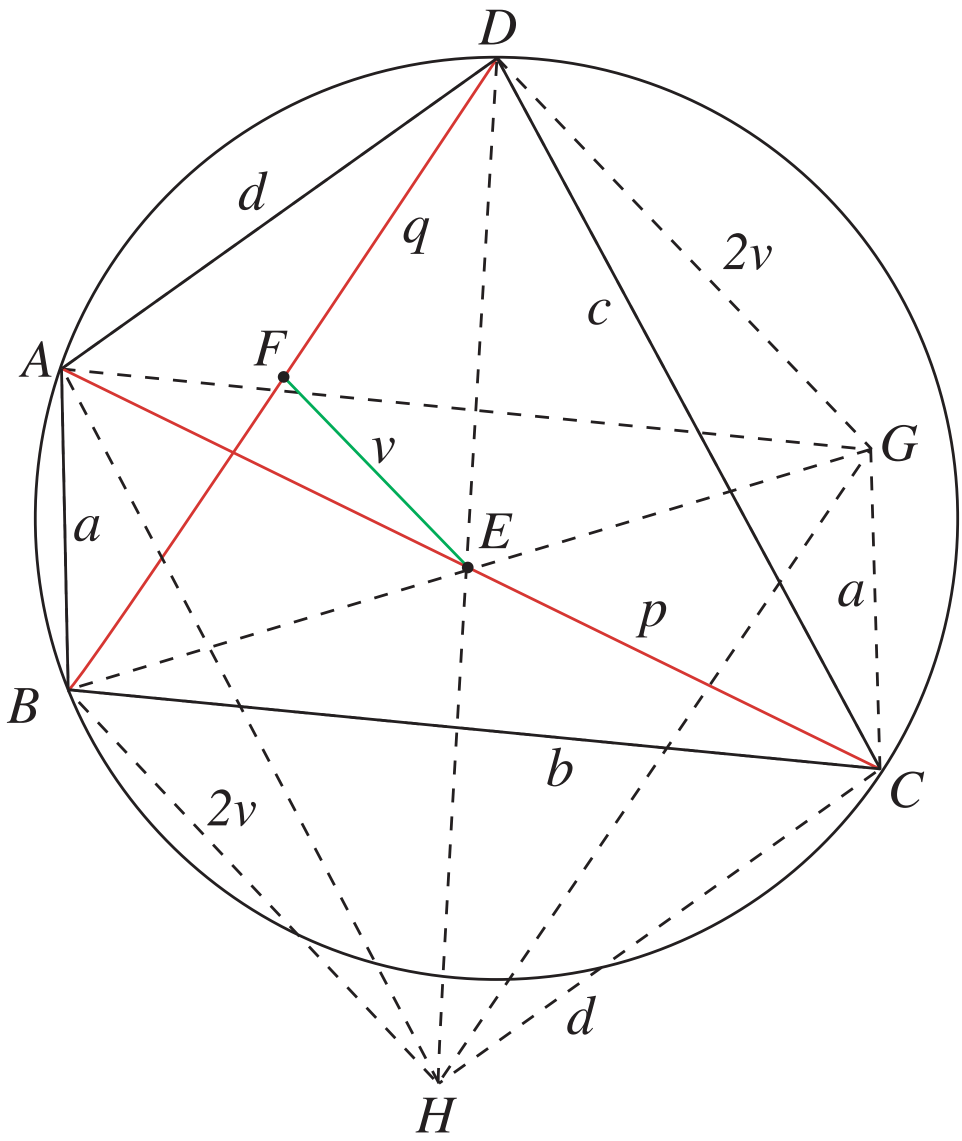

FIGURE 1. $2v > |p - q|$

Theorem 3.3. *For a convex quadrilateral that is cyclic,*

$$pq(p^2 + q^2) > ac(a^2 + c^2) + bd(b^2 + d^2). \tag{5}$$

Proof. Substituting the known cyclic identity $q = \sqrt{\dfrac{(ac + bd)(ab + cd)}{ad + bc}}$ into the leftmost inequality in (4), squaring both sides and simplifying,

$$(ab + cd)(ad + bc) > 4v^2(ac + bd) = (a^2 + b^2 + c^2 + d^2 - p^2 - q^2)(ac + bd),$$

with the equality due to Euler's quadrilateral theorem. The beautiful, symmetrical result follows by expanding the left and right sides, canceling terms, and then substituting pq for $ac+bd$ by Ptolemy's theorem. □

Corollary 3.2. *For a convex quadrilateral that is cyclic,*

$$p^2+q^2>\min\{a^2+c^2,b^2+d^2\}.$$

Proof. Write (5) as $(ac+bd)(p^2+q^2)>ac(a^2+c^2)+bd(b^2+d^2)$. □

Corollary 3.3. *For a convex quadrilateral that is both cyclic and orthodiagonal (its diagonals are perpendicular), $p^2+q^2>4R^2$, where R is the circumradius.*

Proof. Apply Corollary 3.2 to the known identity $4R^2=a^2+c^2=b^2+d^2$ for a quadrilateral that is both cyclic and orthodiagonal. □

Corollary 3.4. *For a convex quadrilateral that is both cyclic and orthodiagonal,*

$$R>v\geq\frac{|p-q|}{2},$$

where R is the circumradius.

Proof. For this case it is known that $R=\sqrt{\dfrac{p^2+q^2+4v^2}{8}}$. Rewrite as $p^2+q^2=8R^2-4v^2$. Applying Corollary 3.3, $8R^2-4v^2>4R^2$, which gives the left side. Applying Proposition 3.1 gives the right side. □

Remark 3.2. *A right kite (which is both cyclic and orthodiagonal) with $p\geq q$ has its Newton line resting on the diagonal of length p. Letting q be arbitrarily small, v will be arbitrarily close in length to R.*

Theorem 3.4. *For a convex quadrilateral that is cyclic, let J be the point at which the extensions of sides a and c meet, let K be the point at which the extensions of sides b and d meet, and let $e=|JK|$. Suppose that $p\neq q$. Then*

$$e>\frac{pq}{p+q}.$$

Proof. It is known that $e|p^2-q^2|=2vpq$, which here we can write as $e=\dfrac{2vpq}{|p^2-q^2|}$. Apply Proposition 3.1. □

4. Bicentric quadrilateral

In this section we consider convex quadrilaterals that are bicentric, that is, both cyclic and tangential. A tangential quadrilateral is one with an incircle. One property of a tangential quadrilateral is that $a+c=b+d$ (the Pitot theorem).

Theorem 4.1. *For P the perimeter of a bicentric quadrilateral,*

$$P^2>8v(p+q)\geq 4|p^2-q^2|. \tag{6}$$

Proof. Substitute $\dfrac{P^2}{4}$ for $(a+c)(b+d)$ in (3). $\square$

This result can be compared to the known inequality $P^2 \geq 8pq$, also for a bicentric quadrilateral. The left inequality in (6) is superior to it when $v \geq \dfrac{pq}{p+q}$, which can be realized, for example, in the case of the right kite (which is bicentric) mentioned in Remark 3.2. In fact, the known inequality and this new inequality can be considered duals of each other, since the known inequality is very tight (in fact equality is attained) for a right kite with equal diagonals (a square) and is very loose for a right kite with one diagonal very small, and vice versa for the new inequality. So we could say the following.

Corollary 4.1. *For P the perimeter of a bicentric quadrilateral, $P^2 \geq \max\{8v(p+q), 8pq\}$. If the max is $8v(p+q)$, the inequality is strict. If v is unavailable, replace $8v(p+q)$ with $4|p^2-q^2|$, and if it is the max, the inequality is strict. In the latter case, $P^2 > 4\max\{p,q\}^2$, obtained from the triangle inequality, offers an even better inequality.*

5. Conclusion

Squaring both sides of a recently strengthened inequality for the perimeter of a quadrilateral leads to the elimination of all squared terms. This good fortune leads to new results, some quite surprising and beautiful. No doubt more results are waiting to be found, and it is hoped that this paper will be an impetus for their discovery.

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