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ON THREE INEQUALITIES INVOLVING THE DISTANCES FROM AN INTERIOR POINT TO THE SIDES OF A TRIANGLE

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Abstract. In this article, using the method of so-called "Difference Substitution" and applying Maple software, the author presents a new proof of a known geometric inequality involving the distances from an interior point to the sides of a triangle. The author also derive two other similar inequalities by using the proved inequality. Two related interesting conjectures are proposed as open problems.

1. Introduction

Let ABC be a triangle with sides $a = BC$, $b = CA$ and $c = AB$. Let P be an arbitrary point of triangle ABC, whose distances to the sides BC, CA and AB are r_1, r_2 and r_3, respectively.

In [1], Chu X.-G. proved the following conjectured inequality posed by Jian Liu in 1997:

$$r_1^2 + r_2^2 + r_3^2 + 8(r_2r_3 + r_3r_1 + r_1r_2) \leq s^2, \tag{1}$$

where $s = (a + b + c)/2$. Motivated and inspired by inequality (1), Chu also proved the following similar inequality:

$$2(r_1^2 + r_2^2 + r_3^2) + 10(r_2r_3 + r_3r_1 + r_1r_2) \leq a^2 + b^2 + c^2. \tag{2}$$

Both inequalities in (1) and (2) hold if and only if $\triangle ABC$ is equilateral and P is its center. In addition, Chu pointed out that the following inequality

$$r_1^2 + r_2^2 + r_3^2 + 11(r_2r_3 + r_3r_1 + r_1r_2) \leq bc + ca + ab \tag{3}$$

holds in [1] and claimed that this inequality can be proved by similar way to prove (1). But he did not give the proof in his paper.

The proofs of (1) and (2) given by Chu are peculiar. It was used the following inequalities respectively (see Lemma 1 and Lemma 2 in [1]):

$$(h_a - r_1)^2 + (h_b - r_2)^2 + (h_c - r_3)^2 \geq 2(r_2r_3 + r_3r_1 + r_1r_2) + \frac{T}{4R^2}, \tag{4}$$

$$(h_a - r_1)^2 + (h_b - r_2)^2 + (h_c - r_3)^2 \geq r_2r_3 + r_3r_1 + r_1r_2 + \frac{K}{4R^2}, \tag{5}$$

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where h_a, h_b, h_c are the length of the altitudes from A, B, C to BC, CA, AB, respectively, R the circumradius of $\triangle ABC$, and

$$T = \frac{-\sum a^4(b^2+c^2) + 2\sum b^3c^3 - 2abc\sum a^3 + 2abc\sum a^2(b+c) + 2(abc)^2}{4\sum bc},$$

$$K = \frac{-\sum a^4(b^2+c^2) + 2\sum b^3c^3 - 4abc\sum a^3 + 4abc\sum a^2(b+c) + 15(abc)^2}{3(\sum a)^2},$$

where $\sum$ denote the cyclic sums over the triple (a, b, c).

On the other hand, we find that Chu did not notice that inequalities (1) and (3) could be easily proved by using (2). The aim of this paper is to give a new proof of inequality (2). Moreover, we shall also derive inequalities (1) and (3) by using (2). Our idea to prove (2) is natural. Indeed, it is easily known that the geometric inequality (2) can be transformed into an algebraic inequality involving three positive numbers and three nonnegative numbers. We consider the proof of this algebraic inequality and then finish the proof of (2).

2. Proof of inequality (2)

Proof. Let S be the area of $\triangle ABC$. By the area relation $S_{\triangle BPC} + S_{\triangle CPA} + S_{\triangle APB} = S$, we have the following identity:

$$ar_1 + br_2 + cr_3 = 2S. \tag{6}$$

Thus, we see that inequality (2) is equivalent to

$$\begin{aligned} &(a^2+b^2+c^2)\,(ar_1+br_2+cr_3)^2 \\ &\geq 4S^2\left[2(r_1^2+r_2^2+r_3^2) + 10(r_2r_3+r_3r_1+r_1r_2)\right]. \end{aligned} \tag{7}$$

Multiplying both sides by 2 and using the equivalent form of Heron's formula:

$$2(b^2c^2+c^2a^2+a^2b^2) - a^4 - b^4 - c^4 = 16S^2, \tag{8}$$

we further know that (7) is equivalent to

$$\begin{aligned} &2(a^2+b^2+c^2)(ar_1+br_2+cr_3)^2 - \big(2b^2c^2+2c^2a^2+2a^2b^2-a^4 \\ &-b^4-c^4\big)\left(r_1^2+r_2^2+r_3^2+5r_2r_3+5r_3r_1+5rr_1r_2\right) \geq 0. \end{aligned} \tag{9}$$

By expanding and arranging, the above inequality becomes the following (required to prove):

$$p_1r_1^2 + p_2r_2^2 + p_3r_3^2 - q_1r_2r_3 - q_2r_3r_1 - q_3r_1r_2 \geq 0, \tag{10}$$

where

$$\begin{aligned} p_1 &= 3a^4 + (b^2-c^2)^2, \\ p_2 &= 3b^4 + (c^2-a^2)^2, \\ p_3 &= 3c^4 + (a^2-b^2)^2, \\ q_1 &= -5a^4 + (10b^2-4bc+10c^2)a^2 - 5(b^2-c^2)^2 - 4bc(b^2+c^2), \\ q_2 &= -5b^4 + (10c^2-4ca+10a^2)b^2 - 5(c^2-a^2)^2 - 4ca(c^2+a^2), \\ q_3 &= -5c^4 + (10a^2-4ab+10b^2)c^2 - 5(a^2-b^2)^2 - 4ab(a^2+b^2). \end{aligned}$$

In fact, we find that inequality (10) holds for all positive real numbers a, b, c and arbitrary nonnegative real numbers r_1, r_2, r_3. In other words, for $u >$

$0, v > 0, w > 0, x \geq 0, y \geq 0, z \geq 0$ (x, y, z not all zero) the following inequality holds:

$$Q_0 \equiv \sum \left[3u^4 + (v^2 - w^2)^2\right] x^2$$
$$+ \sum \left[5u^4 - (10v^2 - 4vw + 10w^2)u^2 + 5(v^2 - w^2)^2 + 4vw(v^2 + w^2)\right] yz$$
$$(11) \geq 0,$$

where $\sum$ denote the cyclic sums over the triples (x, y, z) and (u, v, w).

Next, we shall use "Difference Substitution"(cf.[2-6]) to prove inequality (11). In our original proof, we used a lemma due to L. Yang (see e.g. Lemma 2 in [5]). Later, we obtained the following proof which only applying the discriminant approach of quadratic functions.

By the symmetry, for proving $Q_0 \geq 0$ we may assume that $u \geq v \geq w$ and let

$$(12) \qquad \begin{cases} v = w + p, & (p \geq 0) \\ u = w + p + q & (q \geq 0). \end{cases}$$

Under this hypothesis, the proof of (11) can be divided into the following six cases.

Case 1. The nonnegative x, y, z satisfy $x \geq y \geq z$.

In this case, we set

$$(13) \qquad \begin{cases} y = z + m, & (m \geq 0) \\ x = z + m + n & (n \geq 0). \end{cases}$$

Substituting the equations (12) and (13) into the expression of Q_0 and using the famous software Maple for the calculations, we obtain

$$\begin{aligned} Q_0 = \ & 76w^2pqm^2 + 36w^2pqn^2 + 216wpq^2z^2 + 112wpq^2m^2 + 36wpq^2n^2 \\ & +120wp^2qz^2 + 96wp^2qm^2 + 36wp^2qn^2 + 24w^3pzm + 12w^3pzn \\ & +24w^3pmn + 12w^3qzm + 24w^3qzn + 24w^3qmn + 160w^2p^2zm \\ & +80w^2p^2zn + 76w^2p^2mn + 124w^2q^2zm + 80w^2q^2zn + 58w^2q^2mn \\ & +144wp^3zm + 72wp^3zn + 64wp^3mn + 124wq^3zm + 72wq^3zn \\ & +48wq^3mn + 64p^3qzm + 40p^3qzn + 40p^3qmn + 152p^2q^2zm \\ & +88p^2q^2zn + 68p^2q^2mn + 120pq^3zm + 68pq^3zn + 48pq^3mn \\ & +64wp^3m^2 + 24w^3pm^2 + 12w^3qn^2 + 40pq^3m^2 + 100p^2q^2z^2 \\ & +22w^2p^2n^2 + 84w^2q^2z^2 + 16p^4zn + 56p^2q^2m^2 + 12wq^3n^2 \\ & +18w^2q^2n^2 + 32p^3qm^2 + 32p^4zm + 44w^2q^2m^2 + 40wq^3m^2 \\ & +80wp^3z^2 + 3w^4mn + 76w^2p^2m^2 + 12pq^3n^2 + 12w^3qm^2 \\ & +84w^2p^2z^2 + 32p^3qz^2 + 88wq^3z^2 + 12w^3pn^2 + 18p^2q^2n^2 \\ & +84pq^3z^2 + 12p^3qn^2 + 16wp^3n^2 + 16p^4mn + 28q^4zm \\ & +16q^4zn + 11q^4mn + 160w^2pqzm + 116w^2pqzn + 104w^2pqmn \\ & +320wpq^2zm + 188wpq^2zn + 136wpq^2mn + 216wp^2qzm \\ & +136wp^2qzn + 120wp^2qmn + 3w^4m^2 + 3w^4n^2 + 16p^4z^2 \\ (14) \qquad & +16p^4m^2 + 4p^4n^2 + 20q^4z^2 + 9q^4m^2 + 3q^4n^2 + 84w^2pqz^2 \end{aligned}$$

Since $z \geq 0, p \geq 0, q \geq 0, m \geq 0, n \geq 0$ and $w > 0$, we see that $Q_0 \geq 0$ holds. This completes the proof of $Q_0 \geq 0$ in the first case.

Case 2. The nonnegative x, y, z satisfy $x \geq z \geq y$.
In this case, we set

$$\left\{\begin{array}{ll} z = y + m, & (m \geq 0) \\ x = y + m + n & (n \geq 0). \end{array}\right. \tag{15}$$

Substituting (12) and (15) into (11) and using Maple software, we easily get

$$Q_0 = Q_1 w^2 + P_1, \tag{16}$$

where

$$\begin{aligned}
Q_1 &= 3(n^2 + mn + m^2)w^2 - 12pymw + 2p^2(42y^2 + 13m^2), \\
P_1 &= 16q^4yn + 16p^4yn + 16p^4y^2 + 88wq^3y^2 + 28q^4ym + 84w^2q^2y^2 \\
&\quad +116pq^3ym + 16p^4ym + 188wpq^2yn + 32p^3qy^2 + 124w^2pqym \\
&\quad +116w^2pqyn + 300wpq^2ym + 80wp^3y^2 + 84pq^3y^2 + 100p^2q^2y^2 \\
&\quad +40p^3qym + 136p^2q^2ym + 48w^2pqm^2 + 36w^2pqn^2 + 96wpq^2m^2 \\
&\quad +36wpq^2n^2 + 52wp^2qm^2 + 36wp^2qn^2 + 12w^3pmn + 24w^3qmn \\
&\quad +48w^2p^2mn + 58w^2q^2mn + 40wp^3mn + 48wq^3mn + 24p^3qmn \\
&\quad +56p^2q^2mn + 44pq^3mn + 136wp^2qyn + 20q^4y^2 + 216wpq^2y^2 \\
&\quad +24wp^3m^2 + 12w^3qn^2 + 36pq^3m^2 + 22w^2p^2n^2 + 42p^2q^2m^2 \\
&\quad +12wq^3n^2 + 18w^2q^2n^2 + 12p^3qm^2 + 44w^2q^2m^2 + 40wq^3m^2 \\
&\quad +12pq^3n^2 + 12w^3qm^2 + 12w^3pn^2 + 18p^2q^2n^2 + 12p^3qn^2 \\
&\quad +16wp^3n^2 + 8p^4mn + 11q^4mn + 84w^2pqmn + 124wpq^2mn \\
&\quad +88wp^2qmn + 4p^4m^2 + 4p^4n^2 + 9q^4m^2 + 3q^4n^2 + 84w^2pqy^2 \\
&\quad +120wp^2qy^2 + 160wp^2qym + 68pq^3yn + 12w^3pyn + 40p^3qyn \\
&\quad +24w^3qyn + 12w^3qym + 80w^2p^2yn + 88w^2p^2ym + 88p^2q^2yn \\
&\quad +80w^2q^2yn + 124w^2q^2ym + 72wp^3yn + 88wp^3ym + 72wq^3yn \\
&\quad +124wq^3ym.
\end{aligned}$$

Clearly, $P_1 \geq 0$ holds for $p \geq 0, q \geq 0, m \geq 0, n \geq 0, y \geq 0$ and $w > 0$. It remains to show that $Q_1 \geq 0$. Note that Q_1 is a quadratic function of w and has the following discriminant:

$$\begin{aligned}
F_1 &= -24p^2(42y^2n^2 + 13n^2m^2 + 42y^2mn + 13m^3n \\
&\quad +36y^2m^2 + 13m^4) \leq 0.
\end{aligned} \tag{17}$$

Hence, we conclude that $Q_1 \geq 0$ and then $Q_0 \geq 0$ follows from (15). This completes the proof of inequality (11) in the second case.

Case 3. The nonnegative x, y, z satisfy $y \geq z \geq x$.
In this case, we set

$$\left\{\begin{array}{ll} z = x + m, & (m \geq 0) \\ y = x + m + n & (n \geq 0). \end{array}\right. \tag{18}$$

By substituting (12) and (18) into Q_0, we can obtain the following identity:

$$Q_0 = P_2 + P_3x^2 + N_0x + P_4 + Q_2w^2, \tag{19}$$

where

$$\begin{aligned}
P_2 &= 4(2x+m+n)(2xp+10xw+4xq+pn+pm+6mw \\
&\quad +4nw+nq+mq)p^3, \\
P_3 &= 20q^4+(84p+88w)q^3+(84w^2+216pw+100p^2)q^2 \\
&\quad +12wp(10p+7w)q+84p^2w^2, \\
N_0 &= m\left[(112q^2+104qw+88w^2)p^2+8(q+3w)(3q^2+4qw-w^2)q\right. \\
&\quad \left.+(100q^3+244q^2w+52qw^2-12w^3)p\right]+4n\left[(20qw+20w^2\right. \\
&\quad +16q^2)p^2+(11qw^2+13q^3+3w^3+33q^2w)p \\
&\quad \left.+(q+3w)(3q^2+4qw-w^2)q\right], \\
P_4 &= (7mn+n^2+7m^2)(q+4p+4w)q^3+(22w^2n^2+48w^2mn \\
&\quad +26w^2m^2+32wqmn+12wqn^2+20wqm^2+30q^2m^2 \\
&\quad +32q^2mn+6q^2n^2)p^2+4wp(3w^2mn+3w^2n^2+wqm^2 \\
&\quad +3wqmn+2wqn^2+16q^2m^2+3q^2n^2+17q^2mn), \\
Q_2 &= (3n^2+3m^2+3mn)w^2-12mq(m+n)w \\
&\quad +2q^2(2n^2+11mn+11m^2).
\end{aligned}$$

It is clear that we have $P_2 \geq 0, P_3 \geq 0$ and $P_4 \geq 0$. Since Q_2 is a function of w and has the following discriminant:

$$F_2 = -24q^2(2n^4+13n^3m+18n^2m^2+10m^3n+5m^4) \leq 0. \tag{20}$$

Thus, we know that $Q_2 \geq 0$ holds.

To prove $Q_0 \geq$ in the third case. We may consider two cases $q > w$ and $w \geq q$.

If $q > w$, then we easily see that $N_0 \geq 0$ holds. Since $P_2 \geq 0, P_3 \geq 0$, $P_4 \geq 0$ and $Q_2 \geq 0$, $Q_0 \geq 0$ follows from (19).

If $w \geq q$, then one may put

$$q = w + t. \tag{21}$$

Since $P_2 \geq 0$, we need to prove that

$$P_3x^2 + N_0x + P_4 + Q_2w^2 \geq 0, \tag{22}$$

Substituting (21) into (22), then we easily know that inequality (22) becomes the following:

$$\begin{aligned}
&(3n^2+3mn+3m^2)t^4+(12pn^2+12mpn-24xmq-12xpm+12xpn \\
&-12xnq+12n^2q)t^3+(48pqmn+16xmpq+80xnpq+84x^2pq+44pqn^2 \\
&+4pm^2q+16xq^2m+8xnq^2+84x^2q^2+26p^2m^2+22p^2n^2+84x^2p^2 \\
&+4q^2mn+4q^2m^2+22q^2n^2+48p^2mn+80xnp^2+88xmp^2)t^2 \\
&+8q(16p^2mn+16pqmn+35xmp^2+39xmpq+30xnp^2+32xnpq \\
&+32x^2q^2+3q^2n^2+6q^2m^2+7p^2n^2+9p^2m^2+48x^2pq+36x^2p^2 \\
&+9pm^2q+8pqn^2+26xq^2m+13xnq^2+6q^2mn)t \\
&+4q^2(76x^2p^2+48x^2q^2+96x^2pq+10p^2n^2+28p^2mn+19p^2m^2 \\
&+76xmp^2+48xq^2m+96xmpq+56xnp^2+60xnpq+24xnq^2 \\
&+30pqmn+24pm^2q+9pqn^2+12q^2m^2+12q^2mn+3q^2n^2) \geq 0.
\end{aligned} \tag{23}$$

For proving the above inequality, we have only to prove that

$$\begin{aligned}
&(3n^2+3mn+3m^2)t^2+(12pn^2+12mpn-24xmq-12xpm+12xpn\\
&-12xnq+12n^2q)t+48pqmn+16xmpq+80xnpq+84x^2pq+44pqn^2\\
&+4pm^2q+16xq^2m+8xnq^2+84x^2q^2+26p^2m^2+22p^2n^2+84x^2p^2\\
&+4q^2mn+4q^2m^2+22q^2n^2+48p^2mn+80xnp^2+88xmp^2\geq 0 \qquad (24)
\end{aligned}$$

The left hand side of (24) is a quadratic function of t. With the help of Maple software, we easily obtain its discriminant as follows:

$$\begin{aligned}
F_3 \;=\; &-(2016p^2n^2xm+960pn^3xq+2304m^2p^2nx+816mpn^3q\\
&+432x^2m^2qp+432x^2mq^2n+864xmq^2n^2+1296x^2p^2mn\\
&+1296x^2pn^2q+1152n^2pm^2q+624m^3npq+288m^2nxq^2\\
&+192m^3xpq+1056m^3xp^2+312n^3q^2m+864x^2n^2q^2\\
&+48m^4pq+552p^2n^3m+384xn^3q^2+360n^2q^2m^2\\
&+672p^2n^3x+864x^2p^2m^2+192m^3xq^2+864x^2p^2n^2\\
&+240pn^4q+96m^3nq^2+1008m^2p^2n^2+888m^3np^2\\
&+432x^2m^2q^2+120p^2n^4+120n^4q^2+312m^4p^2\\
&+48m^4q^2+2304pn^2xmq+1728m^2pnxq\\
&+1296x^2mqpn)\leq 0. \qquad (25)
\end{aligned}$$

So, the claimed inequality (24) holds and (22) is proved.

Combining the discussions of the above two cases, we conclude that $Q_0 \geq 0$ holds for nonnegative numbers x, m, n, p, q and positive real number w.

Case 4. The nonnegative x, y, z satisfy $y \geq x \geq z$.

In this case, we set

$$\begin{cases} x = z+m, & (m\geq 0)\\ y = z+m+n & (n\geq 0). \end{cases} \qquad (26)$$

Substituting (12) and (26) into Q_0 and arranging gives the following identity:

$$Q_0 = P_5m^2 + P_6m + 2pP_7 + Q_3, \qquad (27)$$

where

$$\begin{aligned}
P_5 \;=\; &3w^4+24w^3p+12w^3q+76w^2p^2+76w^2pq+44w^2q^2+64wp^3\\
&+96wp^2q+112wpq^2+40wq^3+16p^4+32p^3q+56p^2q^2\\
&+40pq^3+9q^4,\\
P_6 \;=\; &24w^3pz+24w^3pn+12w^3qz+160w^2p^2z+76w^2p^2n\\
&+124w^2q^2z+30w^2q^2n+144wp^3z+64wp^3n+124wq^3z\\
&+32wq^3n+64p^3qz+32pq^3n+44p^2q^2n+152p^2q^2z\\
&+120pq^3z+32p^4z+16p^4n+28q^4z+7q^4n+3w^4n+24p^3qn\\
&+160w^2pqz+48w^2pqn+216wp^2qz+72wp^2qn+320wpq^2z\\
&+88wpq^2n,
\end{aligned}$$

$$\begin{aligned}
P_7 &= 6w^3zn+6w^3n^2+42w^2pz^2+40w^2pzn+11w^2pn^2+42w^2qz^2\\
&+22w^2qzn+4w^2qn^2+40wp^2z^2+36wp^2zn+8wp^2n^2\\
&+60wpqz^2+40wpqzn+6wpqn^2+108wq^2z^2+66wq^2zn\\
&+6wq^2n^2+8p^3z^2+8p^3zn+2p^3n^2+16p^2qz^2+12p^2qzn\\
&+2p^2qn^2+50pq^2z^2+32pq^2zn+3pq^2n^2+42q^3z^2\\
&+26q^3zn+2q^3n^2,\\
Q_3 &= 4(7w+5q)(3w+q)q^2z^2-4(3w+q)(w^2-4wq-3q^2)qnz\\
&+(3w^4+4w^2q^2+4wq^3+q^4)n^2.
\end{aligned}$$

Clearly, inequalities $P_5 \geq 0, P_6 \geq 0$ and $P_7 \geq 0$ hold. By (24), it remains to prove $Q_3 \geq 0$. We consider two cases to finish the proof. If $q > w$, then $Q_3 \geq 0$ is clearly true. If $w \geq q$, denote the quadratic discriminant of Q_3 on z by F_4. A short calculation gives

$$F_4 = 32q^2n^2(w-q)(3w+q)(2q^2+10wq+9w^2)(q+w)^2. \tag{28}$$

So, if $w \geq q$ we see that $F_4 \leq 0$ and then $Q_3 \geq 0$ follows. Above all inequality $Q_3 \geq 0$ is valid for $q \geq 0, n \geq 0, z \geq 0$ and $w > 0$. Thus, inequality $Q_0 \geq 0$ follows from (27).

Case 5. The nonnegative x, y, z satisfy $z \geq x \geq y$.

In this case, we set

$$\begin{cases} x = y+m, & (m \geq 0)\\ z = y+m+n & (n \geq 0). \end{cases} \tag{29}$$

Substituting (12) and (29) into Q_0 and arranging gives

$$Q_0 = P_8 + Q_4w^2, \tag{30}$$

where

$$\begin{aligned}
P_8 &= 52wp^2qm^2+116pq^3ym+40p^3qym+136p^2q^2ym+96wpq^2m^2\\
&+8wpq^2n^2+88wq^3y^2+16p^4ym+12q^4yn+28q^4ym\\
&+32p^3qy^2+84pq^3y^2+80wp^3y^2+36pq^3m^2+100p^2q^2y^2\\
&+12p^3qm^2+24wp^3m^2+40wq^3m^2+4wq^3n^2+42p^2q^2m^2\\
&+4p^2q^2n^2+7q^4mn+4pq^3n^2+300wpq^2ym+16p^4y^2\\
&+216wpq^2y^2+8wp^3mn+32wq^3mn+28p^2q^2mn+28pq^3mn\\
&+24wp^2qyn+20q^4y^2+4p^4m^2+9q^4m^2+q^4n^2+68wpq^2mn\\
&+16wp^2qmn+160wp^2qym+120wp^2qy^2+48pq^3yn\\
&+48p^2q^2yn+16wp^3yn+88wp^3ym+52wq^3yn+112wpq^2yn\\
&+124wq^3ym,\\
Q_4 &= (84y^2+124ym+44yn+44m^2+30mn+4n^2)q^2\\
&+(84py^2+124pym+12wym+8pyn-12wyn+12wm^2\\
&+48pm^2+12pmn)q+(84y^2+88ym+8yn+26m^2\\
&+4mn)p^2-12w(ym+2yn+mn)p+3w^2(m^2+mn+n^2).
\end{aligned}$$

Clearly, we have $P_8 \geq 0$. It remains to prove that $Q_4 \geq 0$. To do this we first show that

$$(84y^2 + 88ym + 8yn + 26m^2 + 4mn)p^2 - 12w(ym + 2yn + mn)p$$
$$+3w^2(m^2 + mn + n^2) \geq 0. \tag{31}$$

The left hand is a quadratic function of p and it is easy to obtain its the discriminant F_5 as follows:

$$F_5 = -24w^2(36y^2m^2 + 18y^2mn + 18y^2n^2 + 44ym^3 + 36ym^2n$$
$$+24ymn^2 + 4yn^3 + 13m^4 + 15m^3n + 9m^2n^2 + 2mn^3) \leq 0. \tag{32}$$

Hence, we deduce that (31) holds.

We now prove $Q_4 \geq 0$. Note that Q_4 is a quadratic function of q and let F_6 be its discriminant. It is easy to verify that

$$\begin{aligned} -F_6 = {} & (21168y^4 + 50400y^3m + 16128y^3n + 43728y^2m^2 + 26880y^2mn \\ & +2688y^2n^2 + 16480ym^3 + 14784ym^2n + 2880ymn^2 + 128yn^3 \\ & +2272m^4 + 2672m^3n + 752m^2n^2 + 64mn^3)p^2 \\ & -96w(63y^3m + 63y^3n + 114y^2m^2 + 159y^2mn + 42y^2n^2 \\ & +65ym^3 + 114ym^2n + 51ymn^2 + 4yn^3 + 12m^4 + 25m^3n \\ & +15m^2n^2 + 2mn^3)p + 24w^2(36y^2m^2 + 54y^2mn + 36y^2n^2 \\ & +50ym^3 + 96ym^2n + 84ymn^2 + 22yn^3 + 16m^4 + 37m^3n \\ & +39m^2n^2 + 17mn^3 + 2n^4). \end{aligned} \tag{33}$$

Clearly, $-F_6$ also is a quadratic function of p and with the positive quadratic term and the constant one. One can compute its discriminant F_7 as follows:

$$\begin{aligned} F_7 = {} & -1536w^2(m^2 + mn + n^2)(2n^2 + 15mn + 22m^2 + 22yn \\ & +62ym + 42y^2)(8yn^3 + 4mn^3 + 120y^2n^2 + 35m^2n^2 \\ & +132ymn^2 + 504y^3n + 876y^2mn + 89m^3n + 492ym^2n \\ & +567y^4 + 466ym^3 + 1227y^2m^2 + 1386y^3m + 64m^4) \leq 0. \end{aligned} \tag{34}$$

Thus, we know that $-F_6 \geq 0$, i.e., $F_6 \leq 0$. Then we conclude that inequality $Q_7 \geq 0$ holds true and the inequality $Q_0 \geq 0$ is proved.

Case 6. The nonnegative x, y, z satisfy $z \geq y \geq x$.

In this case, we set

$$\begin{cases} y = x + m, & (m \geq 0) \\ z = x + m + n & (n \geq 0). \end{cases} \tag{35}$$

Substituting (12) and (35) into Q_0 and arranging gives

$$Q_0 = P_9 + P_{10}w + Q_5w^2, \tag{36}$$

where

$$\begin{aligned}
P_9 &= 20q^4x^2 + 24q^4xm + 12q^4xn + 7q^4m^2 + 7q^4mn + q^4n^2 \\
&+48q^3xpn + 100q^3xpm + 84q^3px^2 + 28q^3pmn + 28q^3pm^2 \\
&+4q^3pn^2 + 24qxp^3m + 32qp^3x^2 + 4qp^3m^2 + 100q^2p^2x^2 \\
&+112q^2xp^2m + 30q^2p^2m^2 + 28q^2p^2mn + 48q^2xp^2n \\
&+4q^2p^2n^2 + 16xp^4m + 16x^2p^4 + 4p^4m^2, \\
P_{10} &= 112q^2xpn + 60q^2pmn + 8q^2pn^2 + 244q^2xpm + 64q^2pm^2 \\
&+216q^2px^2 + 8p^3mn + 80p^3x^2 + 104q^3xm + 88q^3x^2 + 28q^3m^2 \\
&+52q^3xn + 4q^3n^2 + 28q^3mn + 16p^3xn + 24p^3m^2 + 20qp^2m^2 \\
&+8qp^2mn + 120qp^2x^2 + 24qxp^2n + 104qxp^2m + 88xp^3m, \\
Q_5 &= (3mn + 3m^2 + 3n^2)w^2 - (12xpm + 24xpn + 12pmn + 12qmn \\
&+12qxn + 24qxm + 12qm^2)w + 88q^2xm + 84q^2x^2 + 22q^2m^2 \\
&+44q^2xn + 22q^2mn + 4p^2mn + 84qpx^2 + 52qxpm + 8qxpn \\
&+4qpm^2 + 8xp^2n + 84p^2x^2 + 88xp^2m + 26m^2p^2 \\
&+4q^2n^2 - 4pqmn.
\end{aligned}$$

Clearly, $P_9 \geq 0$ and $P_{10} \geq 0$ hold. To prove $Q_0 \geq 0$ it remains to prove that $Q_5 \geq 0$. Note that Q_5 is a quadratic function of w and has positive constant term by $26m^2p^2 + 4q^2n^2 - 4pqmn > 0$. Thus, we only need to prove that its discriminant F_8 is less than or equal to zero. Now, it is easy to obtain

$$\begin{aligned}
-F_8 &= (312m^4 + 432mnx^2 + 576mn^2x + 48n^3m + 864m^2x^2 \\
&+864m^2nx + 216m^2n^2 + 1056m^3x + 360m^3n + 432n^2x^2 \\
&+96n^3x)p^2 + 48q(m^4 - 9mnx^2 - 3mn^2x - n^3m - 6m^2n^2 \\
&-15m^2nx + 9m^2x^2 + 7m^3x - 6m^3n + 2n^3x + 9n^2x^2)p \\
&+24q^2(5m^4 + 18mnx^2 + 13n^3m + 54mn^2x + 30m^2nx \\
&+18m^2x^2 + 18m^2n^2 + 10m^3n + 20m^3x + 36n^2x^2 + 2n^4 + 22n^3x). \qquad (37)
\end{aligned}$$

$-F_8$ can also be regard as quadratic function of p, whose discriminant F_9 is the following:

$$\begin{aligned}
F_9 &= -2304q^2(64m^4 + 466m^3x + 89m^3n + 1227m^2x^2 + 492m^2nx \\
&+35m^2n^2 + 4n^3m + 1386mx^3 + 132mn^2x + 876mnx^2 + 120n^2x^2 \\
&+504nx^3 + 8n^3x + 567x^4)(n^2 + mn + m^2)^2 \leq 0. \qquad (38)
\end{aligned}$$

Therefore, we deduce that $-F_8 \geq 0$, i.e. $F_8 \leq 0$. Hence, inequality $Q_5 \geq 0$ is proved and $Q_0 \geq 0$ follows from the identity (36). This completes the proof in the last case.

Finally, combining the above arguments of six cases, we conclude that inequality (11) holds. The proof of inequality (2) is completed.

Incidentally, it is not difficult to determine that the equality in (11) holds if and only if $x = y = z, u = v = w$. From this we can further know that the equality in (10), (7) and (2) hold if and only if $\triangle ABC$ is equilateral and P is its center.

3. Proofs of inequality (1) and (3)

In this section, we present simple proofs of (1) and (3) by using inequality (2). The facts show that inequality (2) is stronger than (1) and the latter stronger than (3).

Firstly, we apply inequality (2) to derive (1). From (2) we have

$$r_1^2+r_2^2+r_3^2 \le \frac{1}{2}(a^2+b^2+c^2)-5(r_2r_3+r_3r_1+r_1r_2).$$

Thus, to prove (1) we need to prove that

$$\frac{1}{2}(a^2+b^2+c^2)-5(r_2r_3+r_3r_1+r_1r_2) \le s^2-8(r_2r_3+r_3r_1+r_1r_2),$$

which is equivalent to

$$3(r_2r_3+r_3r_1+r_1r_2) \le s^2-\frac{1}{2}(a^2+b^2+c^2). \tag{39}$$

In fact, by a well-known weighted inequality in triangle inequalities (which is equivalent with (3) in [7, p.103]):

$$(xa+yb+zc)^2 \ge 4r(4R+r)(yz+zx+xy), \tag{40}$$

where R, r are the circumradius and inradius of $\triangle ABC$ and x, y, z are arbitrary real numbers. We can immediately obtain the best upper bound of the expression $r_2r_3+r_3r_1+r_1r_2$. Indeed, putting $x=r_1, y=r_2, z=r_3$ in (40) and use previous identity (12), then we get

$$r_2r_3+r_3r_1+r_1r_2 \le \frac{S^2}{(4R+r)r}. \tag{41}$$

Since the equality in (40) holds only when $x:y:z=(b+c-a):(c+a-b):(a+b-c)$, we see that the equality in (41) holds if and only if the barycentric coordinates of P is $(a(b+c-a):b(c+a-b):c(a+b-c))$.

To prove (39), by inequality (41) we have to prove that

$$\frac{3S^2}{(4R+r)r} \le s^2-\frac{1}{2}(a^2+b^2+c^2). \tag{42}$$

In view of (8) and the known identity:

$$s^2-\frac{1}{2}(a^2+b^2+c^2)=(4R+r)r, \tag{43}$$

we see that (42) is equivalent to

$$\left[(a+b+c)^2-2(a^2+b^2+c^2)\right]^2-3(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4) \ge 0.$$

This can be written as

$$2(b+c-a)^2(b-c)^2+2(c+a-b)^2(c-a)^2+2(a+b-c)^2(a-b)^2 \ge 0,$$

which is clearly true. Hence inequalities (42), (39) and (1) are proved.

Finally, we derive inequality (3) by using inequality (1). To prove inequality (3), by (1) we need to prove the following inequality:

$$s^2-8(r_2r_3+r_3r_1+r_1r_2) \le bc+ca+ab-11(r_2r_3+r_3r_1+r_1r_2),$$

or

$$3(r_2r_3+r_3r_1+r_1r_2) \le bc+ca+ab-s^2. \tag{44}$$

But it is easily shown that the right hand of this inequality is equal to that of (39). Consequently, (44) is equivalent with (39). So, inequality (11) is valid.

Remark 3.1. *In [1], the proof of inequality (1) used the following equivalent form of (41):*

$$r_2r_3 + r_3r_1 + r_1r_2 \le \frac{4S^2}{2bc + 2ca + 2ab - a^2 - b^2 - c^2}. \tag{45}$$

However, the author made a mistake by pointing out that the inequality comes from the monograph [7]. In fact, it did not appear in [7]. To the best of my knowledge, inequality (45) is first given by Chen Ji in a Chinese paper [10], where (45) is obtained by an algebraic inequality equivalent with (40).

4. Two open problems

According to the "r-w phenomenon" (see [8] and [9]), we can consider dual inequalities for the proved inequalities involving the distances r_1, r_2, r_3 and other geometric elements. In the following, we propose two related conjectures as open problems.

For inequality (2), we propose the following stronger conjecture after checking by computer.

Conjecture 4.1. *For any interior point P of $\triangle ABC$, we have*

$$2(w_1^2 + w_2^2 + w_3^2) + 10(w_2w_3 + w_3w_1 + w_1w_2) \le a^2 + b^2 + c^2, \tag{46}$$

where w_1, w_2, w_3 are the lengths of the internal bisectors of angles $\angle BPC$, $\angle CPA, \angle APB$, respectively.

Both inequality (39) and (44) are equivalent to

$$r_2r_3 + r_3r_1 + r_1r_2 \le \frac{1}{3}(4R + r)r. \tag{47}$$

In fact, we have the following stronger inequality:

$$r_2r_3 + r_3r_1 + r_1r_2 \le (R + r)r, \tag{48}$$

which can be proved by using (41). We now conjecture that the following further stronger version holds.

Conjecture 4.2. *For any interior point P of $\triangle ABC$, we have*

$$w_2w_3 + w_3w_1 + w_1w_2 \le (R + r)r. \tag{49}$$

If both inequalities (46) and (49) are valid, then the following two inequalities would be proved:

$$w_1^2 + w_2^2 + w_3^2 + 8(w_2w_3 + w_3w_1 + w_1w_2) \le s^2, \tag{50}$$

$$w_1^2 + w_2^2 + w_3^2 + 11(w_2w_3 + w_3w_1 + w_1w_2) \le bc + ca + ab, \tag{51}$$

which are stronger than (1) and (3), respectively.

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