INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 5 (2016), No. 2, 77 - 92

THE FOUR ELLIPSES PROBLEM

GEORGE E. LEFKADITIS, THOMAS L. TOULIAS[1] and
STELIOS MARKATIS

Abstract. Consider three coplanar line segments, having one end point in common, where only two of them are permitted to coincide. Three concentric ellipses can then be defined, say c_i, $i = 1,2,3$, such that every two of these three line segments are considered to be the two conjugate semi–diameters of each ellipse. The present work solves the plane–geometric problem (referred by the authors as the "Four Ellipses" problem) of determining a concentric to c_i ellipse p, circumscribing all c_i, $i = 1,2,3$, using only Synthetic Plane Projective Geometry. G. A. Peschka (1879), in his proof of Karl Pohlke's Fundamental Theorem of Axonometry, solves the above problem through a parallel projection of a sphere onto the c_i's common plane. Therefore, Peschka's methodology (and others) addresses the Four Ellipses problem not as a two–dimensional one but uses the three–dimensional space as a reference space (which the sphere's parallel projection requires). Investigating further the Four Ellipses problem, it is also concluded that the sum of the squares of the three given line segments (which define the three ellipses c_i, $i = 1,2,3$) was found to be equal to the sum of squares of the semi–axes of their circumscribed ellipse p. A series of figures clarify the performed geometric constructions.

1. Introduction

Let e be a plane, embedded in a three–dimensional Euclidean space, $\mathbb{E}^3$. The Karl Pohlke's Theorem, widely known as the Fundamental Theorem of Axonometry[2], is a theorem of the three–dimensional Euclidean space. Specifically, see [4, pg. 250]:

Keywords and phrases: Pohlke's theorem, axonometry, orthogonal homology, conjugate diameters of an ellipse, Rytz construction
(2010)Mathematics Subject Classification: 51N05, 51N15
Received: 23.05.2016. In revised form: 9.09.2016. Accepted: 20.09.2016.

[1]Corresponding author, e-mail: th.toulias@gmail.com

[2]Karl Wilhelm Pohlke (1810–1876), born in Berlin, was a Professor of Descriptive Geometry, and in his book *Darstellende Geometrie*, published in Berlin (1859–1860), includes a Theorem carrying his name, without proof. In 1864, Pohlke's student Hermann Amandus Schwarz (1843–1921), proved this Theorem, which is also known as the Pohlke–Schwarz Theorem. For further details see [8]

Theorem 1.1 (Pohlke). *Consider a bundle of three arbitrary chosen line segments on the Euclidean plane e, say OA, OB and OC, where only one can be of zero length, while points O, A, B and C are not collinear. These segments can always be considered as the parallel projection of three equal and orthogonal to each other line segments in the ambient Euclidean space $\mathbb{E}^3$, say O^*A^*, O^*B^* and O^*C^*. The orthogonal projection is considered as a special case (Fig. 1).*

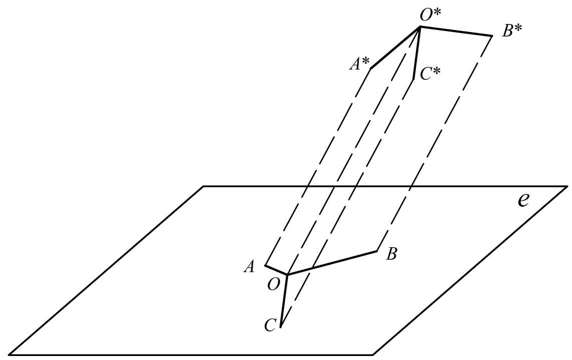

FIGURE 1. Parallel projection for Pohlke's Theorem.

In one of the proofs of this Theorem, a parallel projection of a sphere onto plane e was used; see [5] or [7] for details. In this specific proof, three concentric and coplanar ellipses are considered, say c_1, c_2 and c_3, satisfying the following property by Peschka, [5, pg. 244]:

> *Consider the non–collinear points O,P,Q,R on plane e forming three line segments OA, OB and OC, where two of them can coincide. If the pairs (OB,OC), (OC,OA) and (OA,OB) are considered as the pairs of conjugate semi-diameters of three ellipses c_1, c_2 and c_3 respectively, then a new concentric (to c_i) ellipse exists which circumscribes all c_i, $i=1,2,3$.*

The proof of the above property is derived with the help of an appropriate parallel projection of space $\mathbb{E}^3$ onto plane e, projecting an appropriate sphere S of $\mathbb{E}^3$ onto plane e (on which OA, OB and OC lie)[3]. Under this parallel projection, a cylindrical surface is created tangent to sphere S around a maximum circle $p^* \in S$, which is the sphere's contour resulting from its parallel projection; see Fig. 2. Hence, p^* is parallel–projected onto circumscribing common tangent ellipse p (of the ellipses c_i on plane e), while orthogonal line segments O^*A^*, O^*B^* and O^*C^* are parallel–projected onto conjugate to each othersemi–diameters OA, OB and OC respectively of c_i. According to the above property, these ellipses are defined by pairs of conjugate semi–diameters (OB,OC), (OC,OA) and (OA,OB) respectively, which are the parallel projections of the corresponding maximum circles of the sphere. These maximum circles belong to the planes defined respectively by segments

[3]The use of sphere appears for the first time in a work by J.W.v. Deschwanden and subsequently by G. Peschka in his elementary proof of Pohlke's Fundamental Theorem of Axonometry; see [7].

(O^*B^*, O^*C^*), (O^*C^*, O^*A^*) and (O^*A^*, O^*B^*). Figure 2 demonstrates the above projections method.

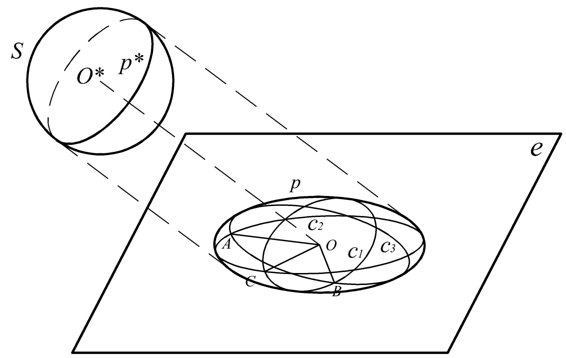

FIGURE 2. Pohlke's Theorem through a parallel–projected sphere S onto plane e.

In principle, the Peschka's property, as described earlier, is a plane–geometric property. The problem now of proving the Peschka's property (i.e. the problem of determining the concentric circumscribing ellipse p of all c_i's), exclusively in terms of Plane Geometry (i.e. without the use of the Euclidean three–dimensional Geometry), shall be investigated in this paper and shall be called hereafter as the *"Four Ellipses"* problem, while p shall be called as a *"common tangential ellipse"* (c.t.e.) of the ellipses c_i, $i = 1,2,3$. A visualization of the Four Ellipses problem is provided in Fig. 3. The derived solution of the Four Ellipses problem is thus considered separately from the well–known Pohlke's Theorem, which provided the initial motivation for this work.

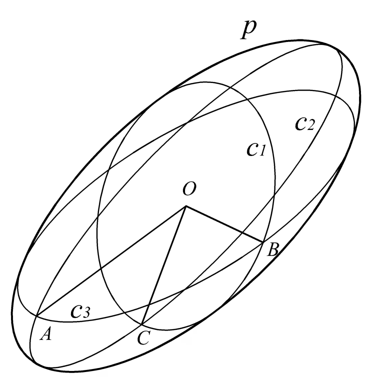

FIGURE 3. The Four Ellipses Problem.

For the derivation of c.t.e. p, plane e is considered to be the Augmented Euclidean Plane, giving us the ability to use Euclidean as well as Projective Geometry's properties and methodologies.

Our approach of proving the existence of c.t.e. p is consisted of three stages: The first two stages, presented in Section 2, two general and useful Lemmas are proved. With their help, the main Theorem addressing the Four Ellipses Problem, is also proved in the third stage in Section 3. Throughout our study we consider, whenever is needed, that the following notions and geometric constructions related to an ellipse, are known, either from the theory of the Plane Projective Geometry, or from the Euclidean Plane Geometry. Particularly, we consider as known:

- The determination of the ellipse's principal axes from a given pair of ellipse's conjugate semi–diameters (Rytz[4] construction); see [6, pg. 69] or [3, pg. 183] for details.
- The notion of the specific orthogonal homology that transforms an ellipse to its secondary (inner) circle, as well as the invariant properties of this transformation.
- The determination of the common points between a line and an ellipse defined by its two principal axes or by its two conjugate semi–diameters.
- The determination of the conjugate of a given diameter of an ellipse, as well the (common) tangent lines of the ellipse at the end points of a diameter.

As far as the figures (in Sections 2 and 3) are concerned, we note that these figures can be considered as an medium of organizing the corresponding logical/geometrical processes, and they do not have any real contribution to our investigation other than providing optical feedbacks; see [2]. We finally point out that each of the three stages, described earlier, can be represented by a single figure. However, for better understanding and clarity of the involved geometric constructions, each of these three figures (depicting the three stages) is broken into a succession sequence of intermediate figures.

2. Useful Lemmas

In this section two Lemmas are presented. Since the proofs of these Lemmas are based on the properties of a projective transformation called orthogonal homology, the plane e —in which our problem is restricted— is considered to be the augmented Euclidean plane.

Lemma 2.1. *Consider an ellipse c' having principal semi–axes OA' and OB of length α and β respectively, $0<\beta<\alpha$. Let E' be the focus corresponding to A', with foci separation (foci semi–distance) $|OE'|=\gamma$. Then, for an arbitrary point L' of the secondary circle $c(O,OB)$ the ellipse c'_1 defined by the pair of conjugate semi–diameters (OE',OL') is tangent to c'; see Fig. 4.*

Proof. From the Projective Geometry it is known that there exists an orthogonal homology, say f, on the plane, mapping ellipse c' to circle c, with axis the line spanned by OB and pair of corresponding points (A',A) under f. If the ellipse c'_1 is tangent internally to ellipse c' —actually there are two diametric contact points for reasons of central symmetry— then the image of c'_1, under homology f, is an ellipse c_1 which should be tangent internally to the circle c. Therefore, it is enough to show that the major semi–axis α_1 of ellipse c_1 equals to circle's radius, i.e. $\alpha_1=\beta$, since this is the only case where an ellipse, inner to a concentric circle, may be tangent to that circle; see Fig. 5.

[4]D. Rytz., Professor at Aarau (1801–1868)

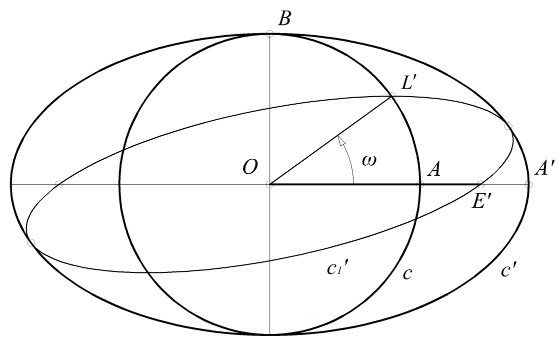

FIGURE 4. Visualization of Lemma 2.1.

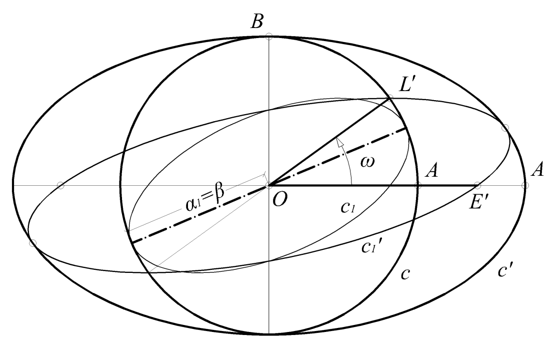

FIGURE 5. The homologue ellipse c_1 of c'_1.

a. Let $\omega := \measuredangle(OA, OL') \in (0, \pi/2)$. The absolute invariant λ of the homology f (depicted in Fig. 6) is given by

(1) $$\lambda := \frac{|OA'|}{|OA|} = \frac{\alpha}{\beta}.$$

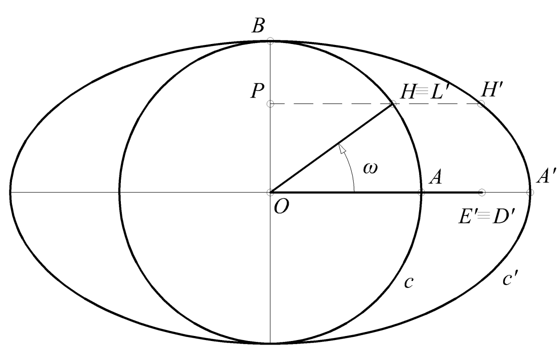

FIGURE 6. The homology f between c' and c.

Thus, for the pair of points (H', H), where $H \equiv L'$ in homology f, it holds that $|PH'|/|PH| = \alpha/\beta$, i.e.

$$|PH'| = |PH|\frac{\alpha}{\beta} = \beta\frac{\alpha}{\beta}\cos\omega = \alpha\cos\omega.$$

We then obtain

$$|PH| = \beta\cos\omega, \tag{2a}$$

$$|OP| = \beta\sin\omega, \tag{2b}$$

$$|PH'| = \alpha\cos\omega. \tag{2c}$$

b. We let, for our convenience in the process of the solution, $E' \equiv D'$. Since (OD', OL') is a pair of conjugate semi–diameters of ellipse c_1', which corresponds, under homology f, to pair (OD, OL) which is a pair of conjugate semi–diameters of ellipse c_1, we consider the following:

b_1. *Calculation of the c_1's requested lengths of OD and OL (Fig. 7).* For point D of the pair (D', D) and for the invariant λ of the homology f it holds that

$$\frac{|OD'|}{|OD|} = \frac{\alpha}{\beta}, \text{ i.e. } |OD| = \frac{\beta\gamma}{\alpha}. \tag{3}$$

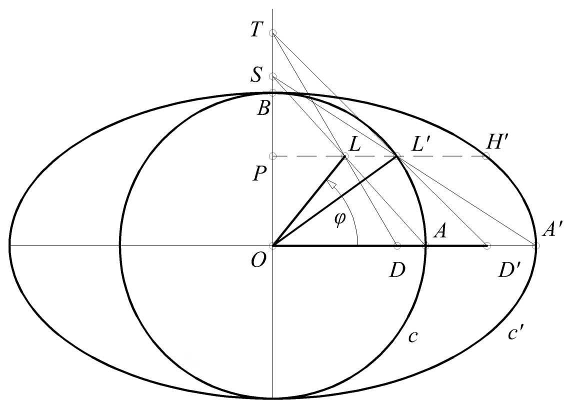

FIGURE 7. The determination of points D and L as homologue points (through f) of D' and L'.

Similarly for point L, the relations (2a)–(2c) yield $|PL'|/|PL| = \alpha/\beta$, i.e.

$$|PL| = |PL'|\frac{\beta}{\alpha} = \frac{\beta^2}{\alpha}\cos\omega. \tag{4}$$

From the orthogonal triangle (OPL) and relation (4) we have

$$|OL|^2 = |OP|^2 + |PL|^2 = \beta^2\sin^2\omega + \left(\frac{\beta^2}{\alpha}\cos\omega\right)^2$$
$$= \beta^2\left(\sin^2\omega + \frac{\beta^2}{\alpha^2}cos^2\omega\right) = \frac{\beta^2}{\alpha^2}\left(\alpha^2\sin^2\omega + \beta^2\cos^2\omega\right),$$

and thus

$$|OL| = \frac{\beta}{\alpha}\sqrt{k}, \tag{5}$$

where $k := \alpha^2\sin^2\omega + \beta^2\cos^2\omega$.

b_2. *Calculation of the angle* $\varphi := \measuredangle(OD, OL) \in (0, \pi/2)$. Relations (2a)–(2c) and (5) imply that

$$\sin\varphi = \frac{|OP|}{|OL|} = \frac{\beta \sin\omega}{\beta\frac{\sqrt{k}}{\alpha}} = \frac{\alpha}{\sqrt{k}}\sin\omega, \tag{6}$$

and finally

$$\sin\varphi = \frac{\alpha}{\sqrt{k}}\sin\omega. \tag{7}$$

b_3. *Calculation of the* c_1*'s major semi–axes length.* Let α_1 and β_1 being the lengths of the c_1's major and minor semi–axis respectively. Setting $\nu := |OD|$ and $\mu := |OL|$, relations (3) and (5) can be written as

$$\nu = \frac{\beta\gamma}{\alpha} \quad \text{and} \quad \mu = \frac{\beta}{\alpha}\sqrt{k}. \tag{8}$$

Recall the Apollonius relations, stating that: *If* (μ, ν) *is an arbitrary pair of conjugate semi–diameters of an ellipse which form an angle* $\varphi \in (0, \pi/2)$ *with each other, then for ellipse's major and minor radius* $0 < \beta_1 < \alpha_1$*, it holds that*

$$\mu^2 + \nu^2 = \alpha_1^2 + \beta_1^2 \quad \textit{and} \tag{9a}$$

$$\mu\nu\sin\varphi = \alpha_1\beta_1. \tag{9b}$$

For further reading see [6] or [3, pg. 178]. Solving the equations (9a) and (9b) with respect to the major semi–axes α_1 we obtain that

$$\alpha_1^2 = \tfrac{1}{2}\left(\mu^2+\nu^2\right) + \tfrac{1}{2}\sqrt{\left(\mu^2+\nu^2\right)^2 - 4\mu^2\nu^2\sin^2\varphi}, \tag{10}$$

where always $\mu^2 + \nu^2 \geq 2\mu\nu\sin\varphi$, and by substitution of μ, ν and $\sin\varphi$ as in (8) and (7), the above relation (10) is then of the form

$$\alpha_1^2 = \tfrac{1}{2}\left(\frac{\beta}{\alpha}\right)^2\left[k+\gamma^2+\sqrt{\left(k+\gamma^2\right)^2 - 4\alpha^2\gamma^2\sin^2\omega}\right]. \tag{11}$$

Taking into account the ellipse's known relation $\gamma^2 = \alpha^2 - \beta^2$, and the fact that $k := \alpha^2\sin^2\omega + \beta^2\cos^2\omega$, the expression (11) finally yields that

$$\alpha_1^2 = \tfrac{1}{2}\left(\frac{\beta}{\alpha}\right)^2\left(2\alpha^2\right) = \beta^2. \tag{12}$$

As the ellipse c_1, which is homologue to the ellipse c_1', adopts $\beta = \alpha_1$ as its major semi–axis, it is therefore a concentric inner tangent ellipse to the circle c; see Fig. 8.

In order to determine the tangent points A_1' and A_2' of ellipses c_1' and c', we first determine the homologue points A_1 and A_2 respectively. Utilizing the Rytz construction, the principal axis of ellipse c_1 are determined; see Fig. 9.

The vertices A_1 and A_2 of c_1, lying on the edges of its major axis of length $2\alpha_1 = 2\beta$, are the requested contact points of the ellipse c_1 and the circle c. Therefore, the ellipse c_1', homologue to c_1 through f, is a (concentric) tangent ellipse to the given c', homologue to the circle c, and inscribed to c'. The corresponding c''s homologue points A_1' and A_2' of points A_1 and A_2 respectively,

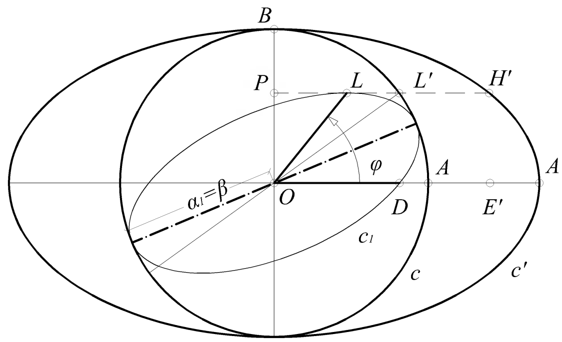

FIGURE 8. The c_1's major semi–axes length $\alpha_1 = \beta$.

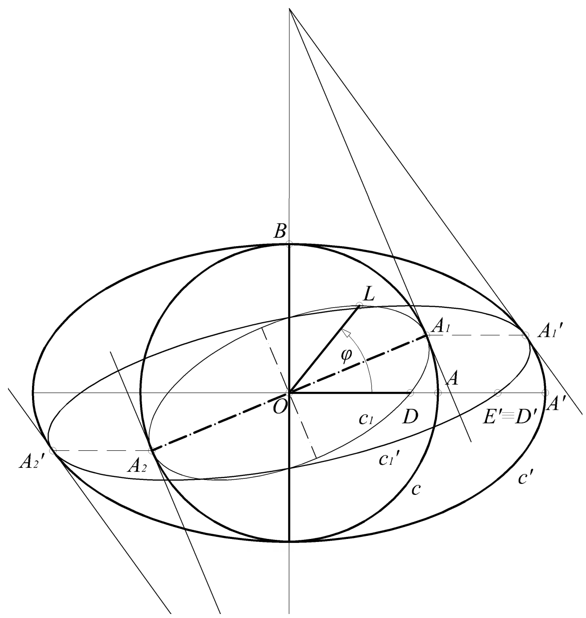

FIGURE 9. The principal axes of the ellipse c_1.

are the requested contact points between c_1' and c'. In these points the common tangent lines are homologue to the corresponding tangent lines at the vertices A_1 and A_2 of the ellipse c_1. The tangent lines at points A_1 and A_2 of c_1 are perpendicular to major axis A_1A_2.

Remark 2.1. *Summarizing the above, we can rewrite Lemma 2.1 as follows: Let circle $c(O,OB)$, a point E' lying on the plane of c and L' any point of circle c. Then, the ellipse c_1', which is defined by the pair of conjugate semi–diameters (OE',OL') (see Fig. 4), is tangent to another ellipse c', which is tangent to given circle c. Ellipse c' has focal point E' and secondary circle the c.*

The following Lemma 2.2 is a generalization and, at the same time, an application of Lemma 2.1.

Lemma 2.2. *Let c be an ellipse with principal axes OA and OB of length α and β respectively, $0 < \beta < \alpha$. Let S be an arbitrary point lying on the plane of c, and H any point of c. Then, the ellipse t which is defined by the pair of conjugate semi–diameters (OS, OH), see Fig. 10, is tangent to another ellipse k at points L_1 and L_2 (Fig. 11), with ellipse k is being tangent to given ellipse c at the edge points of a common diameter M_1M_2. Moreover, the conjugate semi–diameter ON of M_1M_2 lies on the line spanned by OS.*

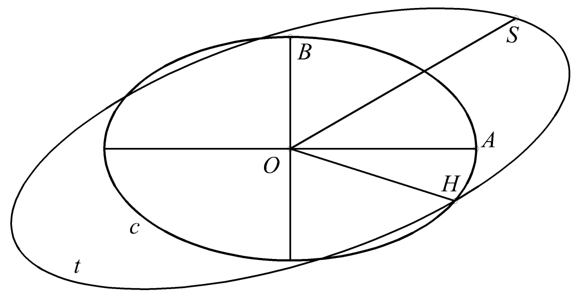

FIGURE 10. The ellipse t defined by its conjugate semi–diameters (OS, OH).

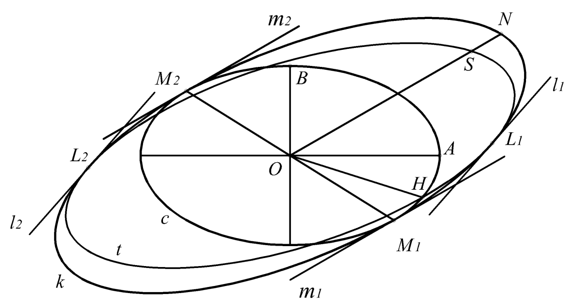

FIGURE 11. The ellipse k tangent to t at their contact point L_1 and L_2.

Proof. Let $c'(O, \beta = |OB|)$ be the secondary circle of given ellipse c (Fig. 12). The orthogonal homology f transforming c to c' is defined with axis OB and a pair of corresponding points (A, A'). Let S' and H' be the corresponding points, under f, of S and H; see Fig. 12.

We construct an ellipse t' defined by the pair of conjugate semi–diameters (OS', OH'); see Fig. 13. Obviously, t' is the homologue ellipse of t, which is defined by the pair of conjugate semi–diameters (OS, OH); see Fig. 10. Therefore, we are reduced to Lemma 2.1, and especially to Remark 2.1. In our case, of lemma 2.2, instead of the circle c we consider now circle $c'(O, \beta = |OB|)$ and instead of point E' we consider point S'. Thus, Lemma 2.2 becomes an application of Lemma 2.1, or its rewritten form as described in Remark 2.1. Due to the above discussion, we can define the requested ellipse k according to the following steps (Fig. 14):

a. We consider the diameter $M_1'M_2'$ of the circle $(O, \beta = |OB|)$ perpendicular to OS'.

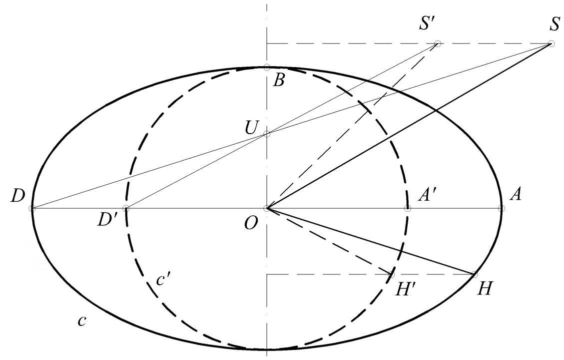

FIGURE 12. The homologue points S' and H' of the S and H (through f).

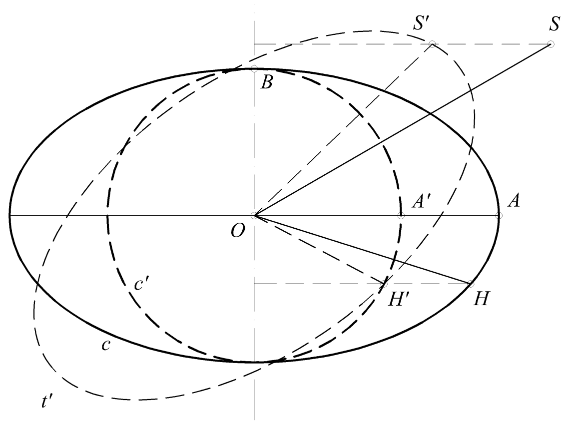

FIGURE 13. The ellipse t' defined by the pair of its conjugate semi–diameters (OS', OH').

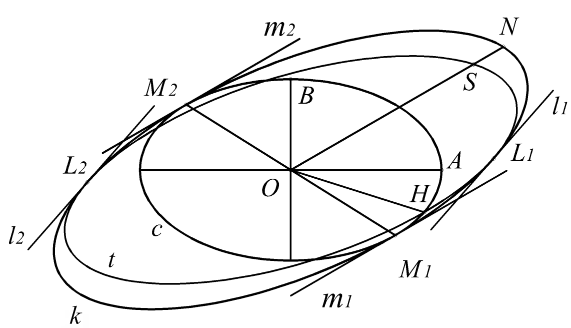

FIGURE 14. Application of Lemma 2.1 with circle c' and given point S'.

b. The ellipse k' is defined having $M_1'M_2'$ as its minor axis and S' as one of its focal points and the length $|M_2'S'|$ of its major semi–axis is defined by the well known relation

$$|OM_2'|^2 + |OS'|^2 = |M_2'S'|^2. \tag{13}$$

c. We let $M_2'S' = ON'$, where ON' is being the major semi–axis of the ellipse k', which is tangent to all corresponding ellipses t' formed as point H' travels over the circle c'. Therefore, as the circle c' is an homologue of the given ellipse c (through f), the requested ellipse k is also an homologue of k' (through the same homology f). It is then sufficient to determine the homologue of points N and M_2 of the k's vertices N' and M_2' respectively, which in turn define a pair (ON, OM_2) of conjugate semi–diameters of the requested ellipse k. It is clear that M_2 is on the given ellipse c, while N lies on the line spanned by OS. Moreover, if OS intersects the ellipse c at point, say W, the homologue OS' also intersects the homologue circle c' at point W'; see Fig. 14.

d. We can then prove that

$$|OS|^2 + |OW|^2 = |ON|^2. \tag{14}$$

With the help of (14), we can easily determine the requested ellipse k. Note that the relation (14) was also proved in [5, pg. 245] with the help of a parallel projection of an appropriate sphere onto a plane. Indeed: Relation (13) can be written as

$$|OW'|^2 + |OS'|^2 = |ON|^2, \tag{15}$$

while

$$\frac{|OW'|}{|OW|} = \frac{|OS'|}{|OS|} = \frac{|ON'|}{|ON|} := h. \tag{16}$$

Thus,

$$|OW'| = h|OW|, \quad |OS'| = h|OS| \text{ and } |ON'| = h|ON|. \tag{17}$$

From (15)–(17) we derive the requested relation (14), which have now proved without the use of projections in a three–dimensional Euclidean space.

In Fig. 15, the pair of the semi–diameters (ON, OM_2) yields, through the corresponding Rytz construction, the principal axes P_1P_2 and R_1R_2 of the requested ellipse k. Since the ellipse k' and the circle c' are tangent to each other on the c''s diametrical points M_1' and M_2', their two common tangent lines m_1' and m_2' at these points respectively, are then also having two common tangent lines m_1 and m_2 of the homologue k and c at their points M_1 and M_2 respectively. Moreover, as the tangent lines m_1' and m_2' are perpendicular to the c''s diameter $M_1'M_2'$, and hence parallel to ON', we conclude that the homologue tangent lines m_1 and m_2 of the tangent lines m_1' and m_2' are also parallel to the line spanned by OS.

Figure 16 presents only the necessary elements that contribute to the better understanding of the previous discussion. Moreover, we point out here that, with the help of (14), we can easily determine the requested ellipse k from points N and M_2 (as we already have mentioned above), i.e. from the pair of its conjugate semi–diameters (ON, OM_2). Indeed, from the orthogonal triangle that is derived through (14), we construct ON dy defining a semi–diameter ON of the requested ellipse k onto OS; see Fig. 16. The parallel to OS tangent lines m_1 and m_2 of the given ellipse c determine the requested

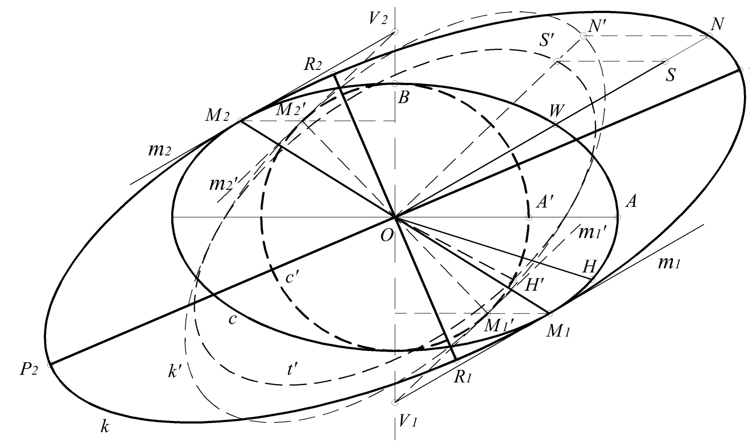

FIGURE 15. The construction of the ellipse k through its semi–diameters' pair (ON,OM_2).

two common contact points M_1 and M_2 between c and k, and hence the requested conjugate semi–diameter (ON,OM_1) of ON.

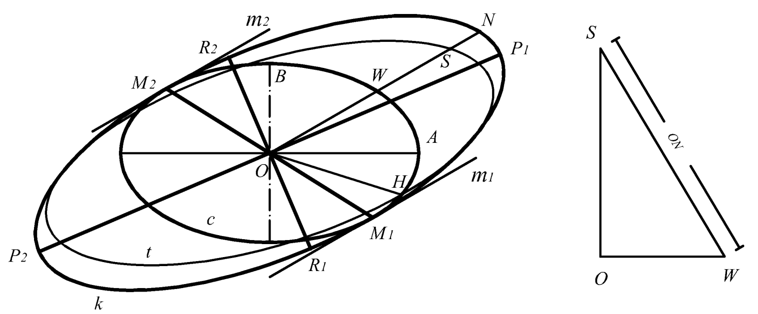

FIGURE 16. The construction of the ellipse k.

In the following steps we determine the contact points L_1 and L_2 as well as the corresponding tangent lines l_1 and l_2 between the ellipse t and its enveloping ellipse k; see Fig. 17.

a. Consider the principal axes P_1P_2 and R_1R_2 of the ellipse k as well as the secondary circle $k''(O,|OR_1|)$. Let f_1 be the orthogonal homology between the ellipses k and k'', with homology axis spanned by R_1R_2 which transforms point P_1 to P_1''. Let also t'' be the homologue of the "variable" ellipse t.
b. For the determination of ellipse t'' we find a pair of its conjugate semi–diameters. Because the homologue t of t'' has a pair of conjugate semi–diameters (OS,OH), the homologue pair (OS'',OH'') is also a pair conjugate semi–diameters of t''. On Fig. 17 points S'' and H'' have been defined through homology f_1. With the Rytz construction we determine the principal axis of t''. From these principal axis the major one is $L_1''L_2''$. Thus, t'' is tangent to k'' at points L_1'' and L_2'', while

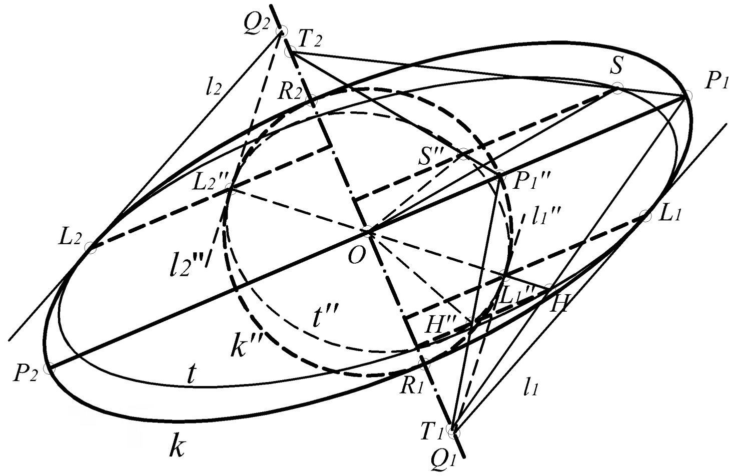

FIGURE 17. The construction of the he contact points L_1 and L_2 as well as the corresponding tangent lines l_1 and l_2 between the tangent ellipses t and k.

the homologue L_1 and L_2 of these points are the requested contact points between ellipses k and t.

c. The two common tangent lines l_1 and l_2 of the ellipse t and k are then the homologue of the common tangent lines l_1'' and l_2'' between the circle k'' and the ellipse t'' at their contact points L_1'' and L_2''.

The proof of Lemma is now complete.

Remark 2.2. *Both Lemmas 2.1 and 2.2 can be rewritten as a single Theorem as follows:* "Let c be an ellipse with principal axes OA and OB of length α and β respectively, $0 < \beta < \alpha$. Let S be an arbitrary point lying on the plane of c, and H any point of c. Then, the ellipse t which is defined by the pair of conjugate semi–diameters (OS, OH) is tangent to another ellipse k, which is tangent to the given ellipse c (see Lemma 2.2)". *The above holds also for the special case where ellipse c is reduced to a circle (see Remark 2.1). Because of the method followed for the proof of both Lemmas, it was essential the special case of the circle to be proved first and then followed by the proof of the ellipse's general case, as the special case was required in the course of the proof of the general case.*

3. THE FOUR ELLIPSES THEOREM

The main result of this study, addressing the Four Ellipses Problem, is derived in this Section with a Theorem, which shall be referred as the *"Four Ellipses"* Theorem.

Main Theorem 3.1 (The Four Ellipses Theorem)**.** *Consider three arbitrary given coplanar line segments OA, OB and OC (non–degenerated), where only two of them can lie on the same line. Each one of the pairs* (OB, OC), (OC, OA) *and* (OA, OB), *is considered to be a pair of conjugate semi–diameters which define the concentric ellipses* c_1, c_2 *and* c_3 *respectively. Then, there always exists an ellipse p, concentric to* c_i, $i = 1,2,3$, *which circumscribes all* c_i *ellipses,* *i.e. being tangent at two points with each one of them. Moreover, if* $0 < \beta < \alpha$ *being the length of the p's principal semi–axes, it holds that*

$$|OA|^2 + |OB|^2 + |OC|^2 = \alpha^2 + \beta^2. \tag{18}$$

Proof. The Four Ellipses Theorem can be proven through the use of Lemma 2.2. We first consider one of the three ellipses, say c_3, defined by the pair of conjugate semi–diameters (OA, OB); see Fig. 18.

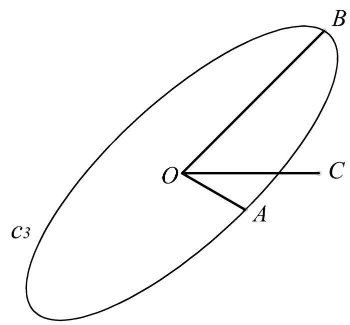

FIGURE 18. The ellipse c_3 defined by the pair of its two conjugate semi–diameters (OA, OB).

Since points A and B belong to the ellipse c_3, applying Lemma 2.2, the concentric ellipses c_1 and c_2 are the tangent ellipses of an ellipse p, which is also tangent to the ellipse c_3; see Fig. 19. Therefore, p is a tangent ellipse to all c_i, $i = 1, 2, 3$.

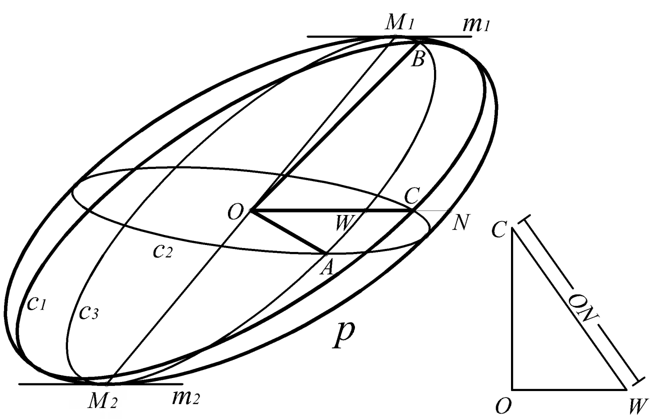

FIGURE 19. The ellipse p being tangent to all c_i, $i = 1, 2, 3$.

Specifically, following Lemma 2.2, we let W be the intersection point between the ellipse c_3, defined by the pair (OA, OB), and the line on which the third given segment OC lies. From the tangent lines m_1 and m_2 of the ellipse c_3 at points M_1 and M_2, which are parallel to OC, we can determine the conjugate to OW diameter M_1M_2 of c_3. An orthogonal triangle COW can then be constructed having as orthogonal sides the given segments OC and OW, while $|ON| = |CW|$l see also Fig. 16. We place ON onto the line spanned by OC and we construct the requested ellipse p having as pair of conjugate semi–diameters the pair (ON, OM_1). The ellipses p and c_3 are clearly tangent to each other at the edge points of their common diameter M_1M_2.

As far as the interesting property (18) is concerned, we consider the following: Recall the first Apollonius relation (9a) in Lemma 2.1, i.e.

$$\mu^2 + \nu^2 = \alpha_1^2 + \beta_1^2 = \text{const.}, \tag{19}$$

where (μ, ν) are being the lengths of any pair of conjugate semi–diameters of a given ellipse, while α_1 and β_1 are correspond to the ellipse's principal semi–axes. Therefore, for the ellipse c_3 and for the two given pairs of conjugate semi–diameters (OA, OB) and (OW, OM_1) it holds that

$$|OA|^2 + |OB|^2 = |OW|^2 + |OM_1|^2. \tag{20}$$

From the orthogonal triangle COW, as constructed above in Fig. 19 through the application of the relation (14) in Lemma 2.2, we obtain that

$$|OC|^2 + |OW|^2 = |ON|^2. \tag{21}$$

Moreover, for the pair (OM_1, ON) of the conjugate semi–diameters of the ellipse p, it holds that

$$|OM_1|^2 + |ON|^2 = \alpha^2 + \beta^2. \tag{22}$$

Adding the relations (20) and (21) and then substituting (22) we finally derive the requested property (18), and hence the Four Ellipses Theorem has been proved.

DISCUSSION

Consider the problem of determining the concentric ellipse p which circumscribes three coplanar and concentric ellipses, say c_i, $i = 1, 2, 3$. Each one of the ellipses c_i is defined by two line segments, which correspond to the ellipse's two conjugate semi–diameters, taken from a bundle of three given line segments on a plane e (assumed that only two of them may coincide). The above problem was addressed be G. A. Peschka as a part of his elementary proof of Pohlke's Fundamental Theorem of Axonometry; see [4, 5, 7]. Although the problem of determining ellipse p is a plane–geometric problem (referred by the authors as the "Four Ellipses" problem), Peschka proved it with the help of a parallel projection of an appropriate sphere onto the c_i's common plane e, which requires the use of a three–dimensional space.

In particular, the ellipses c_i, $i = 1, 2, 3$, are the parallel projections of three great circles of an appropriate sphere S onto plane e, which lies in the ambient three–dimensional space where plane e belongs. The S's three great circles are the intersections of the sphere S with three planes which are perpendicular to each other. Each pair of planes define a diameter of the sphere S. The projection of the three radii, one for each diameter, yields three line segments on plane e, which forms three pairs of conjugate diameters, defining the three given ellipses c_i on plane e.

In the present paper, the Four Ellipses problem was investigated in terms of plane Projective Geometry, and not through the use of the Euclidean three–dimensional space (required for the sphere's parallel projection). Specifically, the problem was addressed through the *"Four Ellipses"* Theorem, presented in Section 3, where the existence of the circumscribing ellipse p was proved.

Moreover, the sum of the squares of the three given line segments (which define the three ellipses $c_i,\ i = 1,2,3$) was found to be equal to the sum of squares of the semi–axes of their circumscribed ellipse p. The provided figures illustrated the corresponding geometric constructions of the proofing process.

References

[1] Emch, A., *Proof of Pohlke's theorem and its generalizations by affinity*, American Journal of Mathematics, **40(4)(1918)** 366–374.

[2] Bunt, L.N.H., Jones, P.S. and Bedient. J.D., *The Historical Roots of Elementary Mathematics*, Dover, 1988.

[3] Ladopoulos, P.D., *Elements of Projective Geometry, Vol. 2* (Greek), Karavia Publ., Athens, 1976.

[4] Lefkaditis, G.E., *Descriptive Methods* (Greek), Athens, 2006.

[5] Müller, E. and Kruppa, E., *Lehrbuch der Darstellenden Geometrie*, Springer–Verlag, Wien, 1961.

[6] Ostermann, A. and Wanner, G., *Geometry by its History*, Springer–Verlag, Berlin Heidelberg, 2012.

[7] Peschka, G.A., *Elementarer Beweis des Pohlke'schen Fundamentalsatzes der Axonometrie* Stzgsb. Math. Nat. Akad. Wien LXXVIII, II Abth., **(1879)** 1043–1054.

[8] Sklenáriková, Z. and Pémová, M., *The Pohlke–Schwarz theorem and its relevancy in the didactics of mathematics*, Quaderni di Ricerca in Didattica (math.unipa.it.), **17(2007)**.

DEPARTMENT OF ARCHITECTURE
PATRAS UNIVERSITY
Rio 26504,PATRAS, GREECE
E-mail address: `glef@upatras.gr`

DEPARTMENT OF INFORMATICS
TECHNOLOGICAL EDUCATIONAL INSTITUTE OF ATHENS
EGALEO 12243, ATHENS, GREECE
E-mail address: `th.toulias@gmail.com`

DEPARTMENT OF MATHEMATICS
NATIONAL TECHNICAL UNIVERSITY OF ATHENS
ZOGRAFOU 15780, ATHENS, GREECE
E-mail address: `markatis@math.ntua.gr`