INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 6 (2017), No. 1, 27 - 35

PORISTIC TRIANGLES OF THE ARBELOS

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Abstract. The arbelos is associated with poristic triangles whose circumcircle is the outer circle forming the arbelos. The triangles give infinitely many sextuplets of Archimedean circles and their intouch triangles share the same nine-point circle, which is also Archimedean and has center on the Schoch line.

1. Introduction

An arbelos is one of the two congruent areas surrounded by three mutually touching circles with collinear centers in the plane in a restricted sense. Circles having common radius equal to the half the harmonic mean of the radii of the two inner circles are said to be Archimedean, which are one of the main topics on the arbelos.

In this article, we show that the arbelos can naturally be associated with a set of poristic triangles, whose circumcircle is the outer circle forming the arbelos. The triangles give two families of infinitely many sextuplets of Archimedean circles. The intouch triangles of the poristic triangles share the same nine-point circle, which is also Archimedean and has center at the point of intersection of the Schoch line and the line passing through the centers of the circles forming the arbelos. Some special cases are considered.

2. Base triangle of the arbelos

In this section we construct a special triangle of the arbelos. For two points P and Q, $P(Q)$ denotes the circle with center P passing through Q, and (PQ) denotes the circle with a diameter PQ. The center of a circle δ is denoted by O_δ.

Let O be a point on the segment AB, and let $\alpha = (AO)$, $\beta = (BO)$ and $\gamma = (AB)$. The configuration of the three circles is denoted by (α, β, γ) and called an arbelos. Let $a = |AO|/2$ and $b = |BO|/2$. Circles of radius $ab/(a+b)$ are called Archimedean circles of (α, β, γ), or said to be Archimedean with

Keywords and phrases: arbelos, Archimedean circles, nine-point circle, Schoch line, poristic triangles
(2010)Mathematics Subject Classification: 51M04, 51N20
Received: 31.07.2016. In revised form: 01.29.2017. Accepted: 14.02.2017.

respect to (α, β, γ), and the common radius is denoted by r_A. We use a rectangular coordinate system with origin O such that the points A and B have coordinates $(2a, 0)$ and $(-2b, 0)$ respectively. The radical axis of α and β is called the axis of the arbelos, which overlaps with the y-axis. The point of intersection of the axis and γ lying in the region $y > 0$ is denote by I. It has coordinates $\left(0, 2\sqrt{ab}\right)$. The external common tangent of α and β touching the two circles in the region $y < 0$ is expressed by the equation [8]:

$$(a-b)x + 2\sqrt{ab}y + 2ab = 0. \tag{1}$$

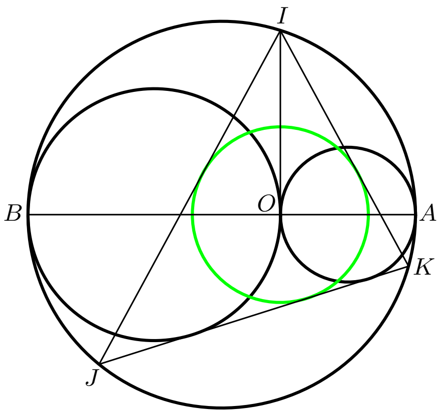

Figure 1.

Let J and K be the points of intersection of γ and this tangent, where J is closer to B than K (see Figure 1). We call IJK the base triangle of (α, β, γ). Let $g = a + b$, $u = a - b$, $w = a^2 + b^2$ and $v = \sqrt{w + ab}$. The points J and K have coordinates

$$\left(\frac{2r_A(u-2v)}{g}, -\frac{2\sqrt{ab}(w-uv)}{g^2}\right) \text{ and } \left(\frac{2r_A(u+2v)}{g}, -\frac{2\sqrt{ab}(w+uv)}{g^2}\right),$$

respectively. Therefore the lines IJ and KI are expressed by the equations

$$vx - \sqrt{ab}y + 2ab = 0 \text{ and } -vx - \sqrt{ab}y + 2ab = 0, \tag{2}$$

respectively. The equations show that the lines IJ and KI are symmetric in the axis. They also show that each of the distances from O to IJ and from O to KI is $2ab/\sqrt{v^2 + ab} = 2r_A$. On the other hand, the distance between O and JK also equals $2r_A$ [2], which is also obtained from (1). Therefore we get:

Theorem 2.1. *The base triangle IJK has incircle of radius $2r_A$ with center O.*

3. Poristic triangles of the arbelos

Let ζ be the incircle of the triangle IJK. Since IJK has circumcircle γ and incircle ζ, there is a continuous family of triangles with the same circumcircle and incircle by the Poncelet closure theorem. We call the triangles the poristic triangles of (α, β, γ). In this section we show that each of the poristic triangles of (α, β, γ) gives several Archimedean circles and the intouch triangles of the poristic triangles share the same nine-point circle.

Let EFG be a poristic triangle of (α, β, γ), and let $E'F'G'$ be its intouch triangle, where E', F' and G' lie on the segments FG, GE and EF, respectively (see Figure 2). Let E_m be the midpoint of EO. The points F_m and G_m are defined similarly.

Theorem 3.1. *The following statements hold.*
(i) *The circles (EO) and (FO) share the chord $G'O$, and the circle $(G'O)$ is Archimedean and is the inverse of the line EF in the circle ζ for the points E, F and G'. Similar facts are true for the points E', F and G and for E, F' and G.*
(ii) *The circle with center E_m touching the sides GE and EF is Archimedean. Similar facts are true for the points F_m and G_m and the corresponding sides of the triangle EFG.*
(iii) *The points E_m, F_m and G_m lie on the circle $(O_\alpha O_\beta)$.*

Proof. The part (i) is obvious. The homothety with center E and ratio $1/2$ carries the circle ζ and the point O into the Archimedean circle touching EF and GE and the point E_m, respectively. This proves (ii). The homothety with center O and ratio $1/2$ carries γ and E into the circle $(O_\alpha O_\beta)$ and E_m. This proves (iii).

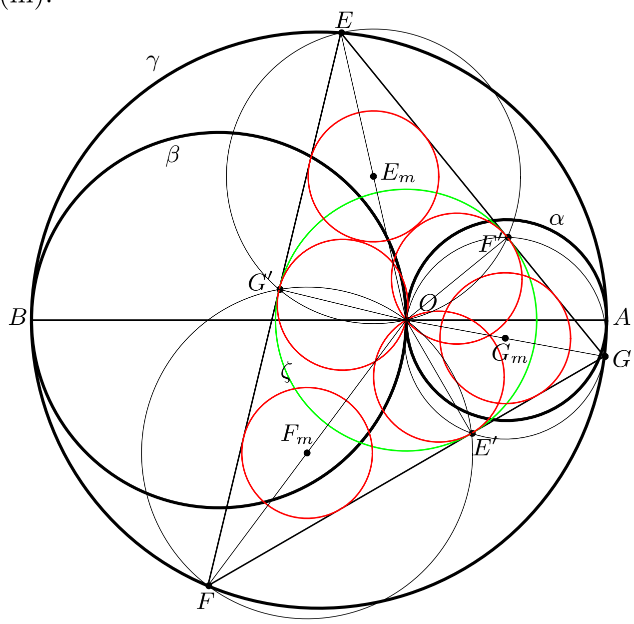

Figure 2.

We get a sextuplet of Archimedean circles from a poristic triangle of (α, β, γ) by the theorem. Therefore we get infinitely many sextuplets of Archimedean circles. For a triangle Δ with inradius $2r$, we can construct an arbelos with Archimedean circles of radius r and a poristic triangle Δ.

Corollary 3.2. *For a triangle Δ with inradius $2r$ and incenter O', let γ' be the circumcircle of Δ. If the line $O'O_{\gamma'}$ intersects γ' at points A' and B', let $\alpha = (A'O')$ and $\beta' = (B'O')$. Then $(\alpha', \beta', \gamma')$ is an arbelos with Archimedean circles of radius r and a poristic triangle Δ.*

Let $s = -r_{\mathrm{A}}u/g$. The line expressed by the equation $x = s$ is called the Schoch line of (α, β, γ) [9]. Let E'' be the point of intersection of $(F'O)$ and

$(G'O)$ different from O. The points F'' and G'' are defined similarly. Let ε be the circle passing through E'', F'' and G'' (see Figure 3). Then ε is Archimedean with respect to (α, β, γ) [4].

Theorem 3.3. *The following statements are true.*
(i) *The point E'' is the midpoint of $F'G'$ and lies on the segment EO. Similar facts are true for the points F'' and G''.*
(ii) *The circle ε is the inverse of γ in the circle ζ and is the nine-point circle of the triangle $E'G'F'$. The center of ε coincides with the point of intersection of the Schoch line and AB.*

Proof. The line $F'G'$ is the inverse of the circle (EO) in ζ, and $EG'OF'$ is a kite. Hence E'' is the midpoint of $F'G'$. This proves (i). Since E'' is the inverse of E in ζ and similar facts are true for F and F'' and for G and G'', the circle ε is the inverse of γ in ζ. The inverses of A and B in ζ are the endpoints of a diameter of ε and have x-coordinates $2r_{\rm A}^2/a$ and $-2r_{\rm A}^2/b$. Hence the center of ε has x-coordinate $r_{\rm A}^2/a - r_{\rm A}^2/b = s$. The rest of (ii) is obvious.

Corollary 3.4. *The intouch triangles of the poristic triangles of (α, β, γ) share the same Archimedean nine-point circle ε. In general, the intouch triangles of the poristic triangles with the same circumcircle and incircle of a triangle share the same nine-point circle.*

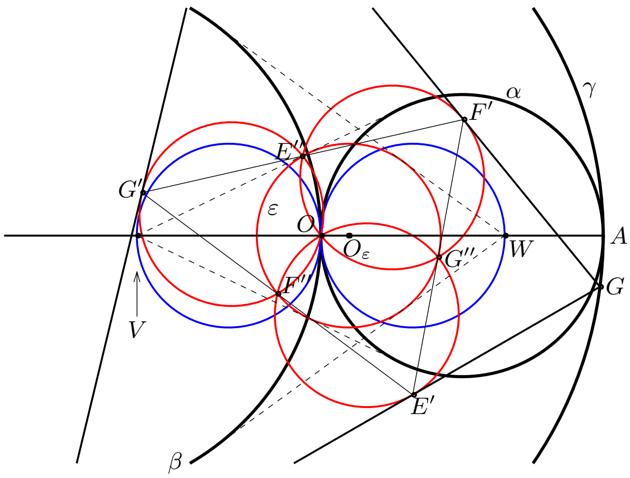

Figure 3.

The theorem creates a new aspect of the Schoch line: it is the perpendicular to AB passing through the center of the inverse of γ in ζ. If $n = -r_{\rm A}/g$, then ε coincides with the circle with center on the Schoch line and touching the two circles with centers $(na, 0)$ and $(nb, 0)$ passing through O internally [9].

Let V (resp. W) be the external center of similitude of the circles ε and α (resp. β). Then the circle (OV) is Archimedean (see Figure 3). For V has coordinates $((-r_{\rm A}a + as)/(a - r_{\rm A}), 0) = (-2r_{\rm A}, 0)$. Similarly (WO) is Archimedean. The circle (WO) (resp. (VO)) is known as an Archimedean circle touching the axis from the side opposite to B (resp. A) and the tangents from A to β (resp. B to α), and is denoted by W_6 (resp. W_7) in

[2]. Therefore the fact creates a new aspect of the Archimedean circles W_6 and W_7.

4. An arbelos derived from Dao's result

From the arbelos (α, β, γ) we can construct another arbelos, which shares Archimedean circles with (α, β, γ). Suppose that the external common tangent of α and β touching the two circles in the region $y > 0$ intersects γ at points S and T (see Figure 4). Let U be the point of intersection of the tangents of γ at S and T. Then $O_\gamma SUT$ is a kite. Hence the circle $(O_\gamma U)$ passes through the points S and T. On the other hand, Dao Thanh Oai has shown that each of the distances from the point I to the two tangents equals $2r_{\mathrm{A}}$ [1]. While the distance between I and the line ST is also $2r_{\mathrm{A}}$ [7]. Hence the triangle STU has incircle of radius $2r_{\mathrm{A}}$ with center I. Therefore if $\alpha' = (O_\gamma I)$, $\beta' = (UI)$ and $\gamma' = (O_\gamma U)$, then the arbelos $(\alpha', \beta', \gamma')$ shares Archimedean circles with (α, β, γ), and STU is a poristic triangle of $(\alpha', \beta', \gamma')$ by Corollary 3.2. The axis of $(\alpha', \beta', \gamma')$ is the tangent of γ at the point I, which is parallel to ST [7]. Let a' and b' be the radii of the circles α' and β', respectively. Then $a' = g/2$. Solving the equation $a'b'/(a'+b') = r_{\mathrm{A}}$ for b', we get $b' = abg/w$. Therefore γ' has radius $g^3/(2w)$.

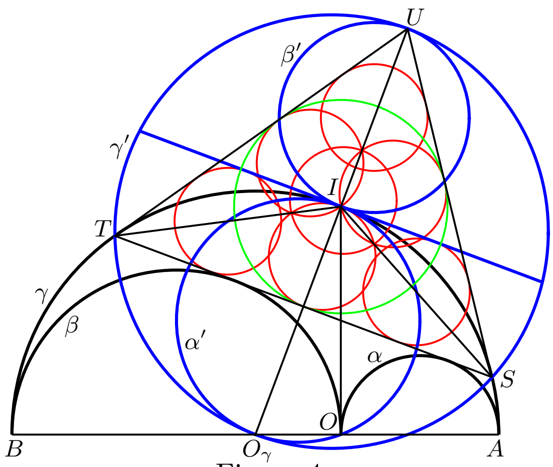

Figure 4.

5. Another infinitely many sextuplets of Archimedean circles

We consider another infinitely many sextuplets of Archimedean circles of (α, β, γ), which are obtained from the following simple fact (see Figure 5):

Proposition 5.1. *If D is a point lying outside a circle $\mathcal{C}$ and M is a point lying on one of the tangents of $\mathcal{C}$ from D, then the distance between M and the line $DO_{\mathcal{C}}$ equals the radius of $\mathcal{C}$ if and only if M lies on the circle $D(O_{\mathcal{C}})$.*

Proof. If N is the foot of perpendicular from $O_{\mathcal{C}}$ to DM, and H is the foot of perpendicular from M to $DO_{\mathcal{C}}$, then the triangles DHM and $DNO_{\mathcal{C}}$ are similar. Hence $|MH| = |O_{\mathcal{C}}N|$ and $|DM| = |DO_{\mathcal{C}}|$ are equivalent.

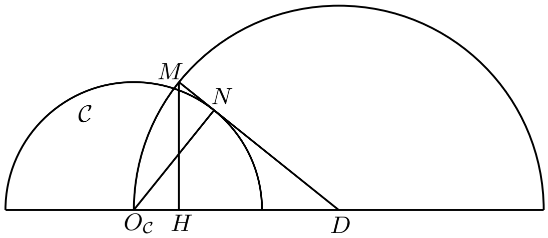

Figure 5.

If each of the lines EF, FG and GE intersects AB, we denote the point of intersections by U_g, U_e and U_f, respectively for a poristic triangle EFG of (α, β, γ). If EF and AB are not parallel, let $\delta_g = U_g(O)$. If EF and AB are parallel, let δ_g be the axis of (α, β, γ). Similarly δ_e and δ_f are defined. By Proposition 5.1, we get the following theorem.

Theorem 5.2. *The smallest circles touching AB and passing through one of the points of intersection of EF and δ_g are Archimedean for a poristic triangle EFG of (α, β, γ). Similar facts are true for FG and δ_e and for GE and δ_f.*

A poristic triangle of (α, β, γ) gives six Archimedean circles in general by Theorem 5.2. Therefore we also get infinitely many sextuplets of Archimedean circles. Figure 6 shows the sextuplet of Archimedean circles in the case in which the poristic triangle is the base triangle.

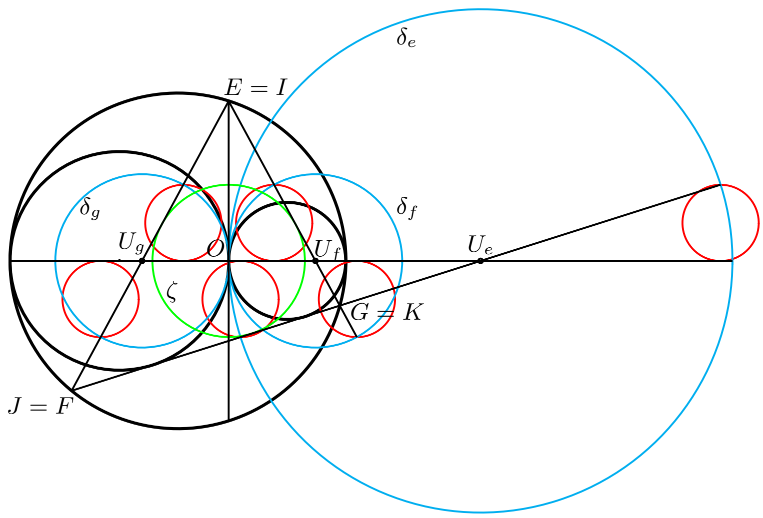

Figure 6.

6. Some special cases

In this section we consider some special cases in which EFG is a special poristic triangle of (α, β, γ). At the beginning, we consider the case the lines AB and FG being parallel (see Figure 7). If $a \neq b$, let $\delta = \delta_e$ in the case $E = I$ (see Figure 6). The circle δ is expressed by the equation $(x+2ab/u)^2 + y^2 = (2ab/u)^2$. If $a = b$, we define δ as the axis. In any case the points of intersection of δ and ζ have coordinates $\left(s, \pm r_{\mathrm{A}}\sqrt{(3a+b)(a+3b)}/g\right)$. Hence they lie on the Schoch line and also on the circle $(O_\alpha O_\beta)$ [6]. Let

Q be the point of intersection of δ and γ lying in the region $y > 0$. Its coordinates are [5]:

$$\left(\frac{-2abu}{w}, \frac{2abg}{w}\right). \tag{3}$$

Let $t = \sqrt{w(w+4ab)}$. The points of intersection of γ and the line $y = -2r_{\rm A}$ have coordinates

$$(u \pm t/g, -2r_{\rm A}). \tag{4}$$

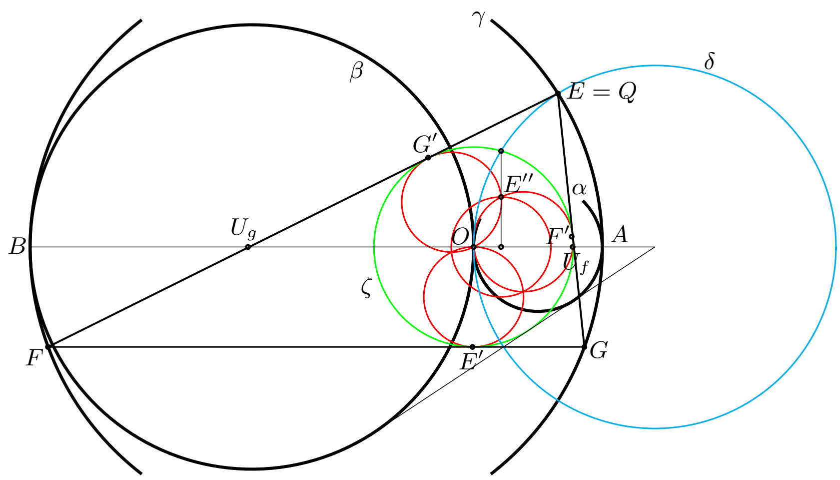

Figure 7.

Theorem 6.1. *The following statements are equivalent for a poristic triangle EFG of (α, β, γ).*
(i) *The lines FG and AB are parallel.*
(ii) *The point E coincides with the point Q or its reflection in AB.*
(iii) *$|FU_g| = |OU_g|$ holds.* (iv) *$|GU_f| = |OU_f|$ holds.*
(v) *The foot of perpendicular from E'' to AB coincides with the point O_ε.*

Proof. Let us assume (i). We may assume FG lies in the region $y < 0$ and F has coordinates $(u - t/g, -2r_{\rm A})$ by (4). Then FQ is expressed by the equation

$$2v^2x + \frac{ug^3 - wt}{2ab}y + gt - uw = 0.$$

The distance between FQ and O equals $2r_{\rm A}$, because we have

$$\frac{(gt-uw)^2}{((2v^2)^2 + ((ug^3 - wt)/(2ab))^2} = 4r_{\rm A}^2.$$

Hence the line FQ touches ζ. Therefore QFG is a poristic triangle of (α, β, γ), i.e., $E = Q$. Therefore (i) implies (ii). Since $E \longmapsto FG$ is one-to-one correspondence, the converse holds. The part (i) is equivalent to that the distance between F and AB is $2r_{\rm A}$. This happens only when $|FU_g| = |OU_g|$ by Proposition 5.1. Hence (i) and (iii) are equivalent. Also (i) and (iv) are equivalent. Let D be one of the farthest points on ε from AB. The slope of the line OQ equals $-g/u$ by (3). Also the slope of the line OD equals $\pm r_{\rm A}/(-r_{\rm A}u/g) = \mp g/u$. Since E'' lies on ε and EO, (ii) holds if and only if E'' coincides with D or its reflection in AB. Hence (ii) and (v) are equivalent.

If FG and AB are parallel, the point of intersection of δ and one of the external common tangents of α and β lies on FG by Proposition 5.1.

Let us assume that E and G are the points of intersection of the circles γ and $A(O)$, where E lies in the region $y > 0$ (see Figure 8). Then EG and the axis are parallel and their distance equals $2r_{\rm A}$ [2], and the triangle EFG is an isosceles triangle, and F coincides with B. In this case the points E' and G' are the points of intersection of the circles β and ζ [9], where the circle $(G'O)$ is denoted by $\mathcal{A}(2)$ in [9]. The distance between the axis and each of the points of intersection of α and $(O_\alpha O_\beta)$ is $r_{\rm A}$ [3]. While the points E_m and G_m lies on $(O_\alpha O_\beta)$ by (iii) of Theorem 3.1, and their distance from the axis also equals $r_{\rm A}$. Therefore E_m and G_m are the points of intersections of α and $(O_\alpha O_\beta)$. The arbelos $(\alpha', \beta', \gamma')$ in Figure 4 is an example of this case.

If $a = b$, then $r_{\rm A} = a/2$. In this case any poristic triangle EFG of (α, β, γ) is equilateral and the circles ζ and $(O_\alpha O_\beta)$ coincide (see Figure 9). Also the circle ε touches the Archimedean circles with centers E_m at the point E''.

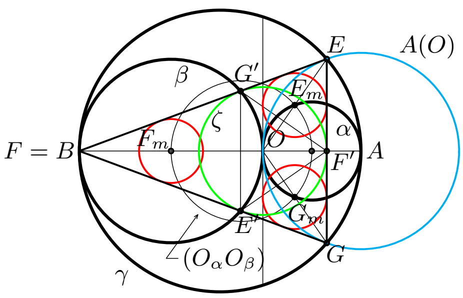

Figure 8.

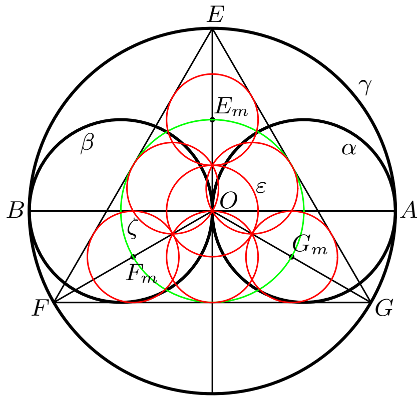

Figure 9.

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