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SPLIT QUATERNIONS and CANAL SURFACES in MINKOWSKI 3-SPACE

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Abstract. A canal surface is the envelope of a one-parameter set of spheres in three-dimensional spaces. In this study, we have defined two canal surfaces by using unit timelike split-quaternions associated with the center curve of the canal surfaces. Also, using the orthogonal matrices corresponding to these unit split-quaternions, these surfaces have been obtained as homotetic motions. Moreover, we have explained the relationship between the types of canal surfaces and unit quaternions.

1. Introduction

Canal surfaces and tube surfaces have been researched intensively in modern differential geometry. Many classifications of canal surfaces have been obtained so far. Canal surface is formed by the envelope of the spheres whose centers lie on the a curve and radius vary depending on this curve [1]. If the radius of canal surface is constant, it is called a tube surface in [1]. Recently, Aslan and Yayli showed that the canal surfaces and tube surfaces can be obtained by the quaternion product and the homothetic motions [11]. Canal surfaces in $\mathbb{E}^3$ given by the different frames (Frenet, Bishop and Darboux frames) were obtained by the same unit quaternion $q(t,\theta) = \cos\theta + \sin\theta T(t)$ [11]. Karacan and Bukcu gave the canal surfaces and tubular surfaces around timelike curve and spacelike curve with spacelike binormal in Minkowski 3- space [6, 7].

Kinematics research motion of the points and lines, velocity and acceleration of any part of some systems. Kinematics are offen referred to motion, which is intensively studied in geometry of motion. Quaternions were discovered in 1843 [12].

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By contribution to kinematics, quaternions have found uses in many modern fields recently. Shoemake described the system of rotation by using quaternions, which is more practical than rotation matrices [2]. Rotation about arbitrary axis is the most important property of quaternions. Many laws in classical and mechanic physics can be easily given by using quaternions. Quaternions are used in many fields of science such as in computer graphics, computer visions and navigation systems.

Ozdemir and Ergin showed that a timelike split quaternion represents a rotation in Minkowski 3-space $\mathbb{E}_1^3$ [8]. Also, they obtained Lorentzian rotation matrix associated with split timelike quaternion. Kula and Yayli showed that the algebra of split quaternions $\widetilde{\mathbb{H}}$ has a scalar product that allows it to identify with the semi-Euclidean space $\mathbb{E}_2^4$. Moreover, they demonstrated that a pair q and p of unit split quaternions in $\widetilde{\mathbb{H}}$ determines a rotation R_{pq} [3]. Tosun et al. investigated some properties of one-parameter homothetic motion in Minkowski 3-space [10].

Babaarslan and Yayli showed that the spacelike constant slope surfaces in Minkowski 3-Space can be obtained by the timelike split quaternions and by the matrix representations [4]. In this paper, by using the unit tangent vector of the center curve of the canal surfaces in Minkowski 3- space, we have defined a unit timelike quaternion with a timelike vector part and a unit timelike quaternion with a spacelike vector part. Then, we have shown that the canal surfaces and the tube surfaces in $\mathbb{E}_1^3$ [7] can be obtained by this unit timelike quaternion with the timelike vector part and the unit normal vector of center curve. Similarly, we have shown that the canal surfaces and the tube surfaces in $\mathbb{E}_1^3$ [6] can be obtained by this unit timelike quaternion with spacelike vector part and the unit binormal vector of center curve. Also, these surfaces have been given by the matrix representation of the unit timelike quaternion and the homothetic motion. After some results were given by the quaternion product and the homothetic motion, it has been given the relationship between the canal surface and the unit quaternion. Finally, examples are given by using the quaternion product and matrices.

2. Preliminaries

Canal surface is formed by the envelope of the spheres whose centers lie on the spine curve $\alpha(t)$ and radius vary depending on the their center curve $\alpha(t)$ [1]. If the radius function $r(t)$ has the property $r'(t) < \|\alpha'(t)\|$, the canal surface is regular. The regular canal surface can be parametrized as

$$C(t,\theta) = \alpha(t) - r(t)r'(t)\frac{\alpha'(t)}{\|\alpha'(t)\|^2} \pm r(t)\frac{\sqrt{\|\alpha'(t)\|^2 - r'(t)^2}}{\|\alpha'(t)\|}(\cos\theta N(t) + \sin\theta B(t)),$$

where $\{T(t), N(t), B(t)\}$ is the Frenet frame of $\alpha(t)$. If the spine curve $\alpha(t)$ is a unit speed curve, the canal surface is parametrized as

$$C(t,\theta) = \alpha(t) - r(t)r'(t)T(t) \pm r(t)\sqrt{1 - r'(t)^2}(\cos\theta N(t) + \sin\theta B(t)).$$

If the radius function $r(t)$ is constant, the canal surface is called tube surface. The tube surface can be given by

$$X(t,\theta) = \alpha(t) + r(\cos\theta N(t) + \sin\theta B(t)),$$

where r is a constant radius [1].

Let $\mathbb{R}^3 = \{x = (x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$ be a three dimensional vector space. For $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, Lorentzian inner product of x and y is given by

$$\langle x, y\rangle_L = -x_1y_1 + x_2y_2 + x_3y_3. \tag{1}$$

$\mathbb{E}_1^3 = \left(\mathbb{R}^3, \langle x, y\rangle_L\right)$ is called Lorentz-Minkowski 3-space. The vector $x \in \mathbb{E}_1^3$ is called a spacelike vector, timelike vector or a null vector, if $\langle x, x\rangle_L > 0$ or $x = 0$, $\langle x, x\rangle_L < 0$ or $\langle x, x\rangle_L = 0$, respectively. The norm of the vector x is $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$. If $\|x\|_L = 1$, it is called a unit vector. Lorentzian vector product of two vectors $x, y \in \mathbb{E}_1^3$ is given by

$$x \wedge_L y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1). \tag{2}$$

For an arbitrary curve $\alpha(s)$ in Lorentz-Minkowski 3-space is locally spacelike, timelike or lightlike, if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or lightlike, for each $s \in I \subset \mathbb{R}$. The vectors $v, w \in \mathbb{E}_1^3$ are orthogonal if and only if $\langle v, w\rangle_L = 0$. Consider a curve $\alpha(s)$ in $\mathbb{E}_1^3$, parameterized by its arc length s. Let $T(s)$ be its tangent vector, i.e., $T(s) = \alpha'(s) = \dfrac{d\alpha(s)}{ds}$. The arc length parameterization of the curve makes $T(s)$ a unit vector, i.e., $\|T(s)\|_L = 1$, therefore its derivative is orthogonal to T. The principal normal vector N is defined as $N = \dfrac{T'}{\|T'\|_L}$. The binormal vector B is defined as the cross product $B = T \wedge_L N$. The Frenet-Serret equations, express the rate of change of the moving orthonormal triad $\{T, N, B\}$ along the curve $\alpha(s)$. The curvature and torsion functions of $\alpha(s)$ can be given, respectively [9] as

$$\kappa = \frac{\|x' \wedge_L x''\|_L}{\|x'\|_L^3}, \tau = \frac{\det(x', x'', x''')}{\|x' \wedge_L x''\|_L^2}.$$

Suppose the center curve of a canal surface is a unit speed spacelike curve with a spacelike binormal

$$\alpha : I \to \mathbb{E}_1^3$$

with nonzero curvature. Then the canal surface can be parameterized

$$C(t,\theta) = \alpha + r(t)r'(t)T(t) \pm r(t)\sqrt{1 - r'(t)^2}\left(N(t)\sinh\theta + B(t)\cosh\theta\right), \tag{3}$$

where T, N and B denote the tangent, principal normal and binormal of α. With the Frenet-Serret system in hand, we can construct a "tubular surface" of radius $r = const.$ about the spacelike curve with a spacelike binormal by defining a surface with parameters t and θ [6]:

$$Tube(t,\theta) = \alpha + r\left(N(t)\sinh\theta + B(t)\cosh\theta\right). \tag{4}$$

Suppose the center curve of a canal surface is a unit speed timelike curve $\alpha : I \to \mathbb{E}_1^3$ with nonzero curvature. Then the canal surface can be parameterized

$$C(t,\theta) = \alpha(t) + r(t)r'(t)T(t) \pm r(t)\sqrt{1 + r'(t)^2}\left(N(t)\cos\theta + B(t)\sin\theta\right), \tag{5}$$

where T, N and B denote the tangent, normal and binormal of α. With the Frenet-Serret system in hand, we can construct a "tubular surface" of radius $r = const.$ about the curve by defining a surface with parameters t and θ [7]:

$$Tube(t,\theta) = \alpha(t) + r\left(N(t)\cos\theta + B(t)\sin\theta\right). \tag{6}$$

Split quaternion algebra is an associative, non-commutative, non-division ring with four basic elements $\{1, i, j, k\}$ satisfying the following equalities

$$i^2 = -1, \;\; j^2 = k^2 = ijk = 1. \tag{7}$$

Scalar and vector parts of split quaternion q are denoted by $S_q = q_1$ and $V_q = q_2 i + q_3 j + q_4 k$, respectively. The split quaternion product $\times_L$ of two quaternions $p = (p_1, p_2, p_3, p_4)$ and $q = (q_1, q_2, q_3, q_4)$ is defined as

$$p \times_L q = p_1 q_1 + \langle V_p, V_q \rangle_L + p_1 V_q + q_1 V_p + V_p \wedge_L V_q, \tag{8}$$

where $\langle , \rangle_L$ and $\wedge_L$ are Lorentzian inner product and vector product, respectively. If $S_q = 0$ then q is called pure split quaternion. The conjugate of a split quaternion is defined as $K_q = S_q - V_q$. Since the vector parts of q and K_q differ only in sign, we have $I_q = q \times_L K_q = K_q \times_L q = q_1^2 + q_2^2 - q_3^2 - q_4^2$. We say that a split quaternion q is spacelike, timelike or lightlike, if $I_q < 0$, $I_q > 0$ or $I_q = 0$ respectively. The norm of $q = (q_1, q_2, q_3, q_4)$ is defined as $N_q = \sqrt{\left|q_1^2 + q_2^2 - q_3^2 - q_4^2\right|}$. If $N_q = 1$, then q is called unit split quaternion [5].

The vector part of any spacelike quaternions is spacelike but the vector part of any timelike quaternion can be spacelike or timelike. Polar forms of the split quaternions are as below

(i) Every unit spacelike quaternion can be written in the form $p = sinh\theta + v\cosh\theta$, where v is a unit spacelike vector in $\mathbb{E}_1^3$.

(ii) Every unit timelike quaternion with the spacelike vector part can be written in the form $p = cosh\theta + v\sinh\theta$, where v is a unit spacelike vector in $\mathbb{E}_1^3$.

(iii) Every unit timelike quaternion with the timelike vector part can be written in the form $p = cos\theta + v\sin\theta$, where v is a unit timelike vector in $\mathbb{E}_1^3$ [3, 8].

Unit timelike quaternions are used to perform rotations in the Minkowski 3-space. If $p = (p_1, p_2, p_3, p_4)$ is a unit timelike quaternion, using the transformation law $\left(p \times_L V_q \times_L p^{-1}\right)_i = \sum_{j=1}^{3} R_{ij} \left(V_q\right)_j$, the corresponding rotation matrix can be found as

$$R_q = \begin{bmatrix} p_1^2 + p_2^2 + p_3^2 + p_4^2 & 2p_1p_4 - 2p_2p_3 & -2p_1p_3 - 2p_2p_4 \\ 2p_2p_3 + 2p_4p_1 & p_1^2 - p_2^2 - p_3^2 + p_4^2 & -2p_3p_4 - 2p_2p_1 \\ 2p_2p_4 - 2p_3p_1 & 2p_2p_1 - 2p_3p_4 & p_1^2 - p_2^2 + p_3^2 - p_4^2 \end{bmatrix}.$$

We can see that all rows of this matrix are orthogonal in the Lorentzian mean [10]. Therefore the unit timelike quaternion $p = (p_1, p_2, p_3, p_4)$ is equivalent to 3×3 orthogonal rotational matrix R_p. The matrix represents a rotation in Minkowski 3-space under the condition that $\det(R_p) = 1$. This is possible with unit timelike quaternions. Also causal character of vector part of the unit timelike quaternion p is important. If the vector part of p is timelike or spacelike then the rotation angle is spherical or hyperbolical, respectively [8].

In Minkowski 3-space, one-parameter homothetic motion of a body is generated by the transformation

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} hA & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix},$$

where $A \in SO_1(3)$, $A^t = \varepsilon A^{-1}\varepsilon$ and the signature matrix ε is the diagonal matrix $(\delta_{ij}\varepsilon_j)$ whose diagonal entries are $\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$. Hence $\varepsilon^{-1} = \varepsilon = \varepsilon^t$. X and X_0 are real matrices of 3×1 type and h is a homothetic scale. A, h and C are differentiable functions of C^∞ class of a parameter t. X and X_0 correspond to the position vectors of the same point with respect to the rectangular coordinate frames of the fixed space $\mathbb{R}$ and the moving space $\mathbb{R}_0$, respectively. At the initial time $t = t_0$, we assume that coordinate systems in $\mathbb{R}$ and $\mathbb{R}_0$ are coincident [10].

3. Canal Surfaces with the unit speed spacelike center curve and Split Quaternions

By using the unit tangent vector of the unit speed spacelike curve $\alpha(t)$, we can obtain a unit timelike quaternion with spacelike vector part $q_1(t,\theta) = \cosh\theta - T(t)\sinh\theta$, where $\alpha : I \to \mathbb{E}_1^3$, $T(t) = (T_1(t), T_2(t), T_3(t))$ is tangent vector and $\langle T(t), T(t)\rangle_L > 0$. We can give rotation matrix R_{q_1} corresponding to the unit timelike quaternion $q_1(t,\theta)$ as

$$R_{q_1} = \begin{bmatrix} \cosh^2\theta + \sinh^2\theta\left(T_1^2 + T_2^2 + T_3^2\right) & -2\sinh\theta\left(\cosh\theta T_3 + \sinh\theta T_1T_2\right) & 2\sinh\theta\left(\cosh\theta T_2 - \sinh\theta T_1T_3\right) \\ 2\sinh\theta\left(\sinh\theta T_1T_2 - \cosh\theta T_3\right) & \cosh^2\theta + \sinh^2\theta\left(-T_1^2 - T_2^2 + T_3^2\right) & 2\sinh\theta\left(\cosh\theta T_1 - \sinh\theta T_2T_3\right) \\ 2\sinh\theta\left(\cosh\theta T_2 + \sinh\theta T_1T_3\right) & -2\sinh\theta\left(\cosh\theta T_1 + \sinh\theta T_2T_3\right) & \cosh^2\theta + \sinh^2\theta\left(-T_1^2 + T_2^2 - T_3^2\right) \end{bmatrix}.$$

For $sp\{T(t)\}$ is the rotation axis of R_{q_1},

$$T(t) = R_{q_1}T(t).$$

Theorem 3.1. *Let unit speed spacelike curve with a spacelike binormal $\alpha(t)$ be the center curve of the canal surface $C(t,\theta)$ in $\mathbb{E}_1^3$, $q_1(t,\theta) = \cosh\theta - T\sinh\theta$ be a unit timelike quaternion with spacelike vector part and $\{T(t), N(t), B(t)\}$ be the Frenet-Serret frame of $\alpha(t)$. By using the quaternion product $q_1(t,\theta)\times_L B(t)$, we can get the canal surfaces and tubular surfaces in $\mathbb{E}_1^3$, whose the center curve $\alpha(t)$ is unit speed spacelike curve with a spacelike binormal as*

$$C(t,\theta) = \alpha(t) + r(t)r'(t)T(t) \pm r(t)\sqrt{1 - r'(t)^2}\,q_1(t,\theta)\times_L B(t), \tag{9}$$
$$Tube(t,\theta) = \alpha(t) + rq_1(t,\theta)\times_L B(t). \tag{10}$$

Proof. By using (8) and (2), for the unit timelike quaternion $q_1(t,\theta) = \cosh\theta - T\sinh\theta$ and the pure quaternion $B(t)$, we get

$$\begin{aligned} q_1(t,\theta)\times_L B(t) &= (\cosh\theta - T\sinh\theta)\times_L B(t) \\ &= (B(t)\cosh\theta - T(t)\wedge_L B(t)\sinh\theta) \\ &= (B(t)\cosh\theta + N(t)\sinh\theta)\,. \end{aligned} \tag{11}$$

If we substitute (11) into (9) and (10), we have (3) and (4) as

$$C(t,\theta) = \alpha_(t) + r(t)r'(t)T(t) \pm r(t)\sqrt{1 - r'(t)^2}\left(B(t)\cosh\theta + N(t)\sinh\theta\right),$$
$$Tube(t,\theta) = \alpha(t) + r\left(N(t)\sinh\theta + B(t)\cosh\theta\right).$$

This completes the proof.

Proposition 3.1. *Let unit speed spacelike curve with a spacelike binormal $\alpha(t)$ be the center curve of the canal surface $C(t,\theta)$ in $\mathbb{E}_1^3$ and the rotation matrix corresponding to the unit timelike quaternion with the spacelike vector part $q_1(t,\theta) = \cosh\theta - T(t)\sinh\theta$ be R_{q_1}. So, by using R_{q_1}, we can give the canal surfaces and tubular surfaces in $\mathbb{E}_1^3$, whose the center curve $\alpha(t)$ is a unit speed spacelike curve with a spacelike binormal, as*

$$C(t,\theta) = \beta(t) + h(t)R_{q_1}B(t),$$
$$Tube(t,\theta) = \alpha(t) + rR_{q_1}B(t),$$

where $\beta(t) = \alpha(t) + r(t)r'(t)T(t)$, $h(t) = \pm r(t)\sqrt{1 - r'(t)^2}$ *and* $B(t)$ *is the binormal vector of* $\alpha(t)$.

Corrolary 3.1. *The canal surfaces in* $\mathbb{E}_1^3$, *whose the center curve* $\alpha(t)$ *is unit speed spacelike curve with a spacelike binormal, can be obtained by the homothetic motion as*

$$C(t,\theta) = \beta(t) + h(t)R_{q_1}B(t),$$

where the position vector of $\beta(t)$ *is the translation vector,* $h(t)$ *is the homothetic scalar and* R_{q_1} *is the orthogonal matrix of homothetic motion.*

Corrolary 3.2. *The tubular surfaces in* $\mathbb{E}_1^3$, *whose the center curve* $\alpha(t)$ *is unit speed spacelike curve with a spacelike binormal, can be obtained by the homothetic motion as*

$$Tube(t,\theta) = \alpha(t) + rR_{q_1}B(t),$$

where the position vector of $\alpha(t)$ *is the translation vector,* r *is the homothetic scalar and* R_{q_1} *is the orthogonal matrix of homothetic motion.*

4. Canal Surfaces with the unit speed timelike center curve and Split Quaternions

Let $q_2(t,\theta) = \cos\theta + T(t)\sin\theta$ be a unit timelike quaternion with timelike vector part, where $T(t)$ is the unit tangent vector of unit speed timelike curve $\alpha : I \to \mathbb{E}_1^3$, $T(t) = (T_1(t), T_2(t), T_3(t))$ and $\langle T(t), T(t)\rangle_L < 0$. We can give rotation matrix R_{q_2} corresponding to the unit timelike quaternion with the timelike vector part $q_2(t,\theta)$ as.

$$R_{q_2} = \begin{bmatrix} \cos^2\theta + \sin^2\theta\,(T_1^2 + T_2^2 + T_3^2) & 2\sin\theta\,(\cos\theta T_3 - \sin\theta T_1T_2) & -2\sin\theta\,(\cos\theta T_2 + \sin\theta T_1T_3) \\ 2\sin\theta\,(\sin\theta T_1T_2 + \cos\theta T_3) & \cos^2\theta + \sin^2\theta\,(-T_1^2 - T_2^2 + T_3^2) & -2\sin\theta\,(\sin\theta T_2T_3 + \cos\theta T_1) \\ 2\sin\theta\,(\sin\theta T_1T_3 - \cos\theta T_2) & 2\sin\theta\,(\cos\theta T_1 - \sin\theta T_2T_3) & \cos^2\theta + \sin^2\theta\,(-T_1^2 + T_2^2 - T_3^2) \end{bmatrix}.$$

For $sp\{T(t)\}$ is the rotaion axis of $q_2(t,\theta)$, we can give the following equation

$$T(t) = R_{q_2}T(t).$$

Theorem 4.1. *Let unit speed timelike curve* $\alpha(t)$ *be the center curve of the canal surface* $C(t,\theta)$ *in* $\mathbb{E}_1^3$, $q_2(t,\theta) = \cos\theta + T(t)\sin\theta$ *be a unit timelike quaternion with timelike vector part and* $\{T(t), N(t), B(t)\}$ *be the Frenet-Serret frame of* $\alpha(t)$. *By using the quaternion product* $q_2(t,\theta) \times_L N(t)$, *we can get the canal surfaces and tubular surfaces in* $\mathbb{E}_1^3$ *whose the center curve*

$\alpha(t)$ is unit speed timelike curve as

$$C(t,\theta) = \alpha(t) + r(t)r'(t)T(t) \pm r(t)\sqrt{1+r'(t)^2}q_2(t,\theta) \times_L N(t), \tag{12}$$
$$Tube(t,\theta) = \alpha(t) + rq_2(t,\theta) \times_L N(t). \tag{13}$$

Proof. By using (8) and (2) for the unit timelike quaternion $q_2(t,\theta) = \cos\theta + T(t)\sin\theta$ and the pure quaternion $N(t)$, we get

$$\begin{aligned} q_2(t,\theta) \times_L N(t) &= (\cos\theta + T(t)\sin\theta) \times_L N(t) \\ &= (N(t)\cos\theta + T(t) \wedge_L N(t)\sin\theta) \\ &= (N(t)\cos\theta + B(t)\sin\theta). \end{aligned} \tag{14}$$

If we substitute (14) into (12) and (13), we have (5) and (6) as

$$C(t,\theta) = \alpha(t) + r(t)r'(t)T(t) \pm r(t)\sqrt{1+r'(t)^2}\,(N(t)\cos\theta + B(t)\sin\theta),$$
$$Tube(t,\theta) = \alpha(t) + r\,(N(t)\cos\theta + B(t)\sin\theta).$$

This completes the proof.

Proposition 4.1. *Let unit speed timelike curve $\alpha(t)$ be the center curve of the canal surface $C(t,\theta)$ in $\mathbb{E}_1^3$ and the rotation matrix corresponding to the unit timelike quaternion with the timelike vector part $q_2(t,\theta) = \cos\theta + T(t)\sin\theta$ be R_{q_2}. So, by using R_{q_2}, we can give the canal surfaces and tubular surfaces in $\mathbb{E}_1^3$, whose the center curve $\alpha(t)$ is a unit speed timelike curve, as*

$$C(t,\theta) = \beta(t) + h(t)R_{q_2}N(t),$$
$$Tube(t,\theta) = \alpha(t) + rR_{q_2}N(t).$$

where $\beta(t) = \alpha(t) + r(t)r'(t)T(t)$, $h(t) = \pm r(t)\sqrt{1+r'(t)^2}$ and $N(t)$ is the normal vector of $\alpha(t)$.

Corrolary 4.1. *The canal surfaces in $\mathbb{E}_1^3$, whose the center curve $\alpha(t)$ is unit speed timelike curve, can be obtained by the homothetic motion as*

$$C(t,\theta) = \beta(t) + h(t)R_{q_2}N(t),$$

where the position vector of $\beta(t)$ is the translation vector, $h(t)$ is the homothetic scalar and R_{q_2} is the orthogonal matrix of homothetic motion.

Corrolary 4.2. *The tubular surfaces in $\mathbb{E}_1^3$, whose the center curve $\alpha(t)$ is unit speed timelike curve, can be obtained by the homothetic motion as*

$$Tube(t,\theta) = \alpha(t) + rR_{q_2}N(t),$$

where the position vector of $\alpha(t)$ is the translation vector, r is the homothetic scalar and R_{q_2} is the orthogonal matrix of homothetic motion.

Remark 4.1. *$q_1(t,\theta)$and $q_2(t,\theta)$ are obtained by unit tangent vector $T(t)$, but they are different timelike quaternions. Canal surfaces [6] are obtanined by the quaternion product of $q_1(t,\theta)$ and binormal vector $B(t)$ as the pure quaternion, but canal surfaces [7] are obtanined by the quaternion product of $q_2(t,\theta)$ and normal vector $N(t)$ as the pure quaternion. So, we can say that these two types of surfaces have different unit timelike quaternions q_1 and q_2, and are obtained by different normal vectors $B(t)$ and $N(t)$.*

Remark 4.2. *Canal surfaces [6] and unit timelike quaternion with spacelike vector part $q_1(t,\theta)$ are defined by hyperbolical angle. Similarly, canal surfaces [7] and unit timelike quaternion with timelike vector part $q_2(t,\theta)$ are defined by spherical angle. So, it can be said that there is a close relationship between types of the canal surface and the unit quaternion.*

Example 1. For a unit speed spacelike curve with spacelike binormal $\alpha(t) = (\cosh\left(\frac{t}{\sqrt{2}}\right), \sinh\left(\frac{t}{\sqrt{2}}\right), \frac{t}{\sqrt{2}})$, the Frenet frame vectors can be given as

$$\begin{aligned}
T(t) &= (\frac{1}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\cosh\left(\frac{t}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}),\\
N(t) &= (\cosh\left(\frac{t}{\sqrt{2}}\right), \sinh\left(\frac{t}{\sqrt{2}}\right), 0),\\
B(t) &= (\frac{1}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\cosh\left(\frac{t}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}).
\end{aligned}$$

So, we can give the rotation matrix R_{q_1} corresponding to the unit timelike quaternion with the spacelike vector part $q_1(t,\theta) = \cosh\theta - T(t)\sinh\theta$, $\beta(t) = \frac{t}{\sqrt{2}}$ as

$$\begin{bmatrix}
\cosh^2\theta + \frac{\sinh^2\theta}{2}\left(\sinh^2\beta + \cosh^2\beta + 1\right) & -\sinh\theta\left(\sqrt{2}\cosh\theta + \sinh\theta\sinh\beta\cosh\beta\right) & \sinh\theta\left(\sqrt{2}\cosh\theta\cosh\beta - \sinh\theta\sinh\beta\right)\\
\sinh\theta\left(\sinh\theta\sinh\beta\cosh\beta - \sqrt{2}\cosh\theta\right) & \cosh^2\theta + \frac{\sinh^2\theta}{2}\left(1 - \sinh^2\beta - \cosh^2\beta\right) & \sinh\theta\left(\sqrt{2}\cosh\theta\sinh\beta - \sinh\theta\cosh\beta\right)\\
\sinh\theta\left(\sqrt{2}\cosh\theta\cosh\beta + \sinh\theta\sinh\beta\right) & -\sinh\theta\left(\sqrt{2}\cosh\theta\sinh\beta + \sinh\theta\cosh\beta\right) & \cosh^2\theta + \frac{\sinh^2\theta}{2}\left(\cosh^2\beta - \sinh^2\beta - 1\right)
\end{bmatrix}.$$

If we use Corollary 3.1. for R_{q_1} and $B(t) = (\frac{1}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\cosh\left(\frac{t}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}})$,

$$\begin{aligned}
Tube(t,\theta) &= \alpha(t) + rR_{q_1}B(t),\\
&= (\cosh\left(\frac{t}{\sqrt{2}}\right) + r\cosh\left(\frac{t}{\sqrt{2}}\right)\sinh\theta + r\frac{1}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right)\cosh\theta,\\
&\sinh\left(\frac{t}{\sqrt{2}}\right) + r\sinh\left(\frac{t}{\sqrt{2}}\right)\sinh\theta + r\frac{1}{\sqrt{2}}\cosh\left(\frac{t}{\sqrt{2}}\right)\cosh\theta,\\
&\frac{t}{\sqrt{2}} - r\frac{1}{\sqrt{2}}\cosh\theta).
\end{aligned}$$

For $r = 5$, we can draw Figure 1.

Example 2. For unit speed timelike curve $\alpha(t) = (\frac{\sqrt{5}t}{2}, \sin\frac{t}{2}, \cos\frac{t}{2})$, the Frenet frame vectors can be given as

$$\begin{aligned} T(t) &= (\frac{\sqrt{5}}{2}, \frac{1}{2}\cos\frac{t}{2}, -\frac{1}{2}\sin\frac{t}{2}), \\ N(t) &= (0, -\sin\frac{t}{2}, -\cos\frac{t}{2}), \\ B(t) &= (\frac{1}{2}, \frac{\sqrt{5}}{2}\cos\frac{t}{2}, -\frac{\sqrt{5}}{2}\sin\frac{t}{2}). \end{aligned}$$

So, for the unit timelike quaternion with timelike vector part $q_2(t,\theta) = \cos\theta + T(t)\sin\theta$ and the normal vector $N(t) = (0, -\sin\frac{t}{2}, -\cos\frac{t}{2})$, if we use (13), we can have tubular surfaces as

$$\begin{aligned} Tube(t,\theta) &= \alpha(t) + rq_2(t,\theta) \times_L N(t) \\ &= \alpha(t) + r\,(N(t)\cos\theta + B(t)\sin\theta) \\ &= (\frac{\sqrt{5}t}{2} + \frac{1}{2}r\sin\theta, \sin\frac{t}{2} - r\cos\theta\sin\frac{t}{2} + \frac{\sqrt{5}}{2}r\sin\theta\cos\frac{t}{2}, \\ &\cos\frac{t}{2} - r\cos\theta\cos\frac{t}{2} - \frac{\sqrt{5}}{2}r\sin\theta\sin\frac{t}{2}). \end{aligned}$$

For $r = \frac{1}{2}$, we can draw Figure 2.

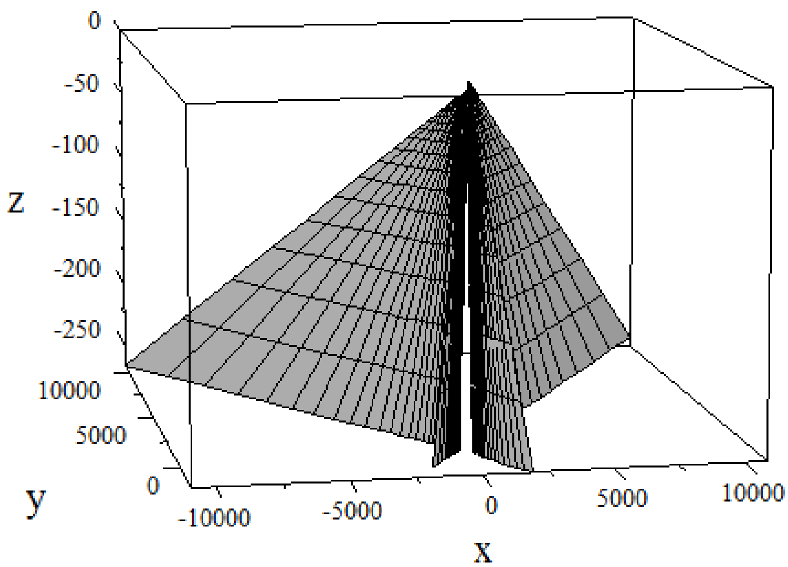

Fig. 1. Tube surface around spacelike curve from Example 1

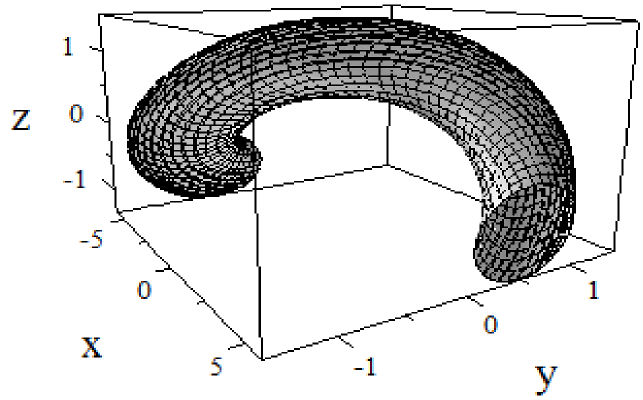

Fig. 2. Tube surface around timelike curve from Example 2

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