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THE ERDÖS - MORDELL THEOREM IN THE EXTERIOR DOMAIN

PETER WALKER

Abstract. We show that in the Erdős-Mordell theorem, the part of the region in which the inequality holds, and which lies outside the triangle, is bounded if and only if the sum of the sines of the two smaller angles is strictly greater than 3/2.

1. Introduction and Examples

The Erdős-Mordell theorem states that if P is an interior or boundary point of the triangle ABC, and X, Y, Z are the feet of the altitudes from P to the sides BC, CA, AB (produced if necessary), then

$$PA + PB + PC \geq 2\,(PX + PY + PZ)\,. \tag{1}$$

There is equality here if and only if the triangle is equilateral and P is its centre. For short, elegant proofs see [1], [2].

For points outside the triangle, the question of the signs of the lengths of the altitudes PX etc. becomes relevant. More precisely, if we take the sign of PX as positive when P is on the same side of BC as the vertex A, otherwise negative, and similarly for the other altitudes, then an extension of (1) is valid for all points in the plane. [B. Malešević et. al., arxiv.org/ftp/arxiv/papers/1204.1003.pdf].

In this paper we consider instead the situation in which we use the absolute distance for the altitudes, that is the lengths PX etc. are considered positive in all cases. These are the *metrical trilinear* coordinates, see Somerville, Analytical Conics, page 157 [ia700705.us.archive.org/16/items/AnalyticalConics/Somerville-AnalyticalConics.pdf].

Since the inequality is strict on the boundary of the triangle, there must be some closed set T say, containing the boundary of the triangle in its interior, on which the inequality remains true, and a complementary open set F say, on which it fails. Our main aim is to to investigate the boundedness or otherwise of the set T; theorem 3.1 below gives a necessary and sufficient condition.

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We begin by looking at the natural special cases of equilateral and of isosceles right-angled triangles. Consider first an equilateral triangle ABC of side 1 and consider what happens when P is on BC produced, at a distance x from C. The altitudes from P to BC, CA, AB have lengths 0, $x\sqrt{3}/2$ and $(x+1)\sqrt{3}/2$ respectively, while the distances from P to the vertices A, B, C are $\sqrt{x^2+x+1}$, $x+1$ and x respectively (the first from the cosine rule since $\cos 120° = -1/2$).

Hence the inequality to be satisfied is

$$\begin{aligned} \sqrt{x^2+x+1} + 2x + 1 &\geq 2\,(2x+1)\,\sqrt{3}/2, \text{ or} \\ \sqrt{x^2+x+1} &\geq (2x+1)\left(\sqrt{3}-1\right). \end{aligned}$$

Substituting $y = x + 1/2$ we easily get

$$x \leq \sqrt{\frac{3}{4\left(15-8\sqrt{3}\right)}} - \frac{1}{2} = 0.30983..$$

is required for P to be in T; beyond this P is in F. The same result is of course valid by symmetry on all other sides, making us surmise that T is bounded in this case, and we shall see later that this is true.

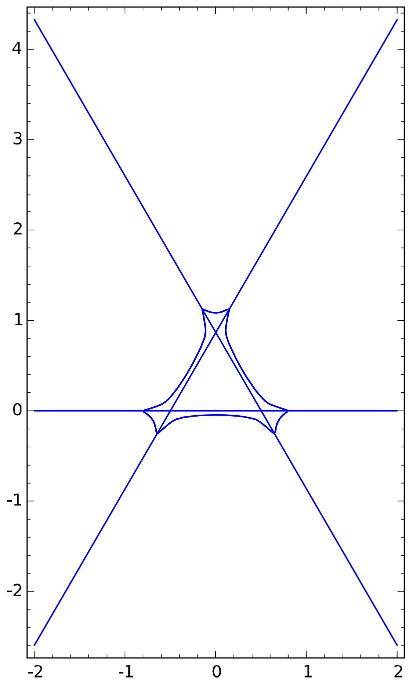

Figure 1

Now consider an isosceles right-angled triangle with the right angle at A and sides $1, 1, \sqrt{2}$. Let P be a point on AB produced at a distance y from B. The altitudes from P to BC, CA, AB have lengths respectively $y/\sqrt{2}$, $y+1, 0$ respectively, while the distances from P to the vertices A, B, C are $y+1, y$ and $\sqrt{y^2+2y+2}$, respectively (the latter from the cosine rule with

$\cos 135° = -1/\sqrt{2}$). Hence the inequality to be satisfied is

$$\begin{aligned}\sqrt{y^2+2y+2}+2y+1 &\geq 2\left(\left(1+1/\sqrt{2}\right)y+1\right), \text{ or} \\ \sqrt{y^2+2y+2} &\geq \sqrt{2}y+1.\end{aligned}$$

From this we find

$$y \leq \sqrt{4-2\sqrt{2}}-2\sqrt{2}+1 = 0.68817...$$

is required for P to be in T; beyond this P is in F.

A similar calculation in which P lies on BA produced at a distance z from A shows that $z \leq 2-\sqrt{2} = 0.58578...$ is required for P to be in T, beyond this P is in F.

Suppose finally that P is on the hypotenuse BC produced at a distance t from C. The altitudes from P to BC, CA, AB have lengths respectively 0,$t/\sqrt{2}$, $\left(t+\sqrt{2}\right)/\sqrt{2}$ respectively, while the distances from P to the vertices A, B, C are $\sqrt{t^2+\sqrt{2}t+1}$, $t+\sqrt{2}$, and t, respectively. Hence the inequality to be satisfied is

$$\begin{aligned}\sqrt{t^2+\sqrt{2}t+1}+2t+\sqrt{2} &\geq 2\left(\sqrt{2}t+1\right), \text{ or} \\ \sqrt{t^2+\sqrt{2}t+1} &\geq \left(\sqrt{2}-1\right)\left(2t+\sqrt{2}\right).\end{aligned}$$

Solving this gives the requirement that P should be in T to be

$$\left(8\sqrt{2}-11\right)\left(t^2+\sqrt{2}t\right)+\left(4\sqrt{2}-5\right) \geq 0.$$

But $8\sqrt{2}-11$ and $4\sqrt{2}-5$ are both positive so this condition is satisfied for all $t \geq 0$ and so T contains the whole of the side BC extended to infinity.

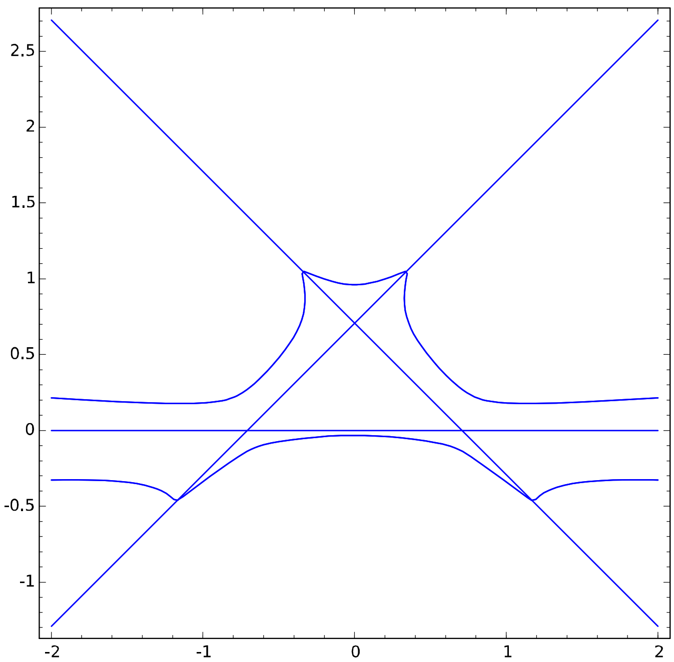

Figure 2

These examples show that T may be bounded or unbounded; the remainder of this paper investigates when these cases occur.

2. Behaviour at Infinity

Consider what happens as $P \to \infty$ in a given fixed direction. This is equivalent to fixing P and considering the special case in which the triangle shrinks to a point while the sides maintain fixed directions. More precisely, suppose that our triangle ABC is positively oriented with the vertices labelled so that the angles α, β, γ at A, B, C are in decreasing order of magnitude, or equivalently the lengths of the sides, or sines of the angles are in decreasing order. (This equivalemce is trivial by the sine rule except when $\alpha > 90°$ when it follows from the cosine rule!) Now fix a line l, extended to infinity in both directions, inclined at an angle θ anticlockwise to the side AB. (It turns out that the exact point at which l intersects AB is irrelevant.) Then, as already noted, instead of thinking of the point P going to infinity along l, we may regard P as fixed while the triangle shrinks to a single point D say, which also lies on l. Note that the sides of the triangle are inclined at angles α, γ, β in anticlockwise order around D. Hence $PA + PB + PC$ will become $3PD$ while PX, PY, PZ will become $PD\left|\sin(\beta+\theta)\right|, PD\left|\sin(\theta)\right|,$ $PD\left|\sin(\alpha-\theta)\right|$ respectively, the absolute values occurring since all sides are taken positively.

Lemma 2.1. *Let* $f(\theta) = \left|\sin(\beta+\theta)\right| + \left|\sin(\alpha-\theta)\right| + \left|\sin(\theta)\right|.$ *Then for a point* P *on* l *sufficiently far from the triangle,* P *will be in* T *or* F *according to whether* $f(\theta) < 3/2$ *or* $f(\theta) > 3/2$.

Proof. We want

$$\begin{aligned} PA + PB + PC &> 2(PX + PY + PZ), \text{or in the limit} \\ 3PD &> 2\left(PD\left|\sin(\beta+\theta)\right| + PD\left|\sin(\alpha-\theta)\right| + PD\left|\sin(\theta)\right|\right) \end{aligned}$$

from which the result is immediate.

The examples in section 1 verify these conclusions in the special cases considered there. The case of equality will be investigated in section 3.

Since the graph of $|\sin|$ is concave downwards on all intervals of the form $(n\pi, (n+1)\pi)$ with local minima at $n\pi$ at which the derivative is discontinuous, we see that the graph of f has a discontinuous derivative where $\theta = -\beta, 0, \alpha$ and is concave downwards elsewhere. We have $f(-\beta) = \sin(\beta) + \sin(\gamma)$, $f(0) = \sin(\beta) + \sin(\alpha)$, $f(\alpha) = \sin(\alpha) + \sin(\gamma)$ and since we are assuming $\alpha \geq \beta \geq \gamma$ it follows that $f(-\beta) \leq f(\alpha) \leq f(0)$. In particular $f(-\beta) = \sin(\beta) + \sin(\gamma)$ is the global minimum of f and the following result follows at once from Lemma 2.1.

Corollary 2.1. *Let* P, l *be as in Lemma 2.1.*

(i) If $\sin(\beta) + \sin(\gamma) > 3/2$ *then a straight line in any direction will eventually lie in* F.

(ii) If $\sin(\beta) + \sin(\gamma) < 3/2$ *then any line parallel to the side* BC *will eventually lie in* T.

Corollary 2.2. *For any triangle, any line parallel to the angle bisector at* A *(the largest angle) will eventually lie in* F. *In particular,* F *is never empty.*

Proof. We shall show that

$$f\,(\alpha/2) = 2\sin(\alpha/2) + |\sin(\beta+\alpha/2)| \le 3/2 \tag{2}$$

is impossible. For if (2) holds then $\sin(\alpha/2) \le 3/4$ and so $\alpha/2 \le \sin^{-1}(3/4) = 48.590°$, $\alpha \le 97.181°$. Also since α is the largest angle, we have $\alpha \ge 60°$, $\sin(\alpha/2) \ge 1/2$. In addition $\beta \le \alpha$ so $30° < \beta + \alpha/2 \le 3\alpha/2 \le 145.77° < 150°$ and $\sin(\beta+\alpha/2) > 1/2$. This contradicts (2) as required.

Note that the property of being bounded in any direction is not sufficient to show that a set is bounded; consider for instance the case of the parabolic arc given by $y = x^2$. Hence Corollary 2.1(i) does not immediately imply that T is bounded. To remove this uncertainty we prove

Theorem 2.3. *Let $\alpha \ge \beta \ge \gamma$ be the angles of a triangle ABC. Then T is bounded if $sin\beta + \sin\gamma > 3/2$.*

More explicitly, suppose that $\sin\beta + \sin\gamma - 3/2 = \delta > 0$. Let E be the union of 3 circular discs centred at the vertices of the triangle, and of radius $3S/2\delta$ where S is the perimeter $a+b+c$. Then $T \subseteq E$.

Proof. Consider points P which lie on a line which passes through A in a direction between the sides AB and AC. P may lie on either side of A, but we suppose initially that P is on the same side of A as the side BC. Let Q be the intersection of PA with BC.

For the distances of P to the vertices we have

$$\begin{aligned} PA+PB+PC &< PA+(PA+AB)+(PA+AC) \\ &< 3PA+S. \end{aligned}$$

For the altitudes we have

$$\begin{aligned} PX+PY+PZ &= PQ\,|\sin(\beta+\theta)| + PA\,|\sin(\alpha-\theta)| + PA\,|\sin(\theta)| \\ &> (PA-S)\,|\sin(\beta+\theta)| + PA\,|\sin(\alpha-\theta)| + PA\,|\sin(\theta)| \\ &> PAf\,(\theta) - S \ge PA\,(3/2+\delta) - S \end{aligned}$$

where $PQ > PA - S$ follows since for any triangle the distance from a vertex to a point on the opposite side is less than the longest side of the triangle and hence less than S.

Here $2(PA\,(3/2+\delta) - S) > 3PA + S$ is true if $2PA\delta > 3S$, and so (1) is negated for $PA > 3S/2\delta$. The argument if P lies on the opposite side of a is slightly simpler since we need only $PQ > PA$ in this case, and the argument for other vertices is similar.

3. The Case of Equality

We begin by describing the possible cases in which $\sin\beta + \sin\gamma = 3/2$. Two particular isosceles cases are of interest. The most obvious case is when $\sin\beta = \sin\gamma = 3/4$, when $\sin\alpha = \sin(\beta+\gamma) = 3\sqrt{7}/8$. We denote these values by $\alpha_0, \beta_0, \gamma_0$ respectively. Numerically $\beta_0 = \gamma_0 = 48.590°$ and $\alpha_0 = 82.819°$. The other case is when $\gamma < \beta = \alpha$. We then have $\sin\beta + \sin\gamma = 3/2$ and $\sin(2\beta) - \sin\gamma = 0$. Eliminating $\sin\gamma$ results in a fourth degree equation in $\sin\beta$:

$$4\sin^4\beta - 3\sin^2\beta - 3\sin\beta + 9/4 = 0.$$

This may be solved either numerically or algebraically, giving $\beta = \alpha = 73.642°$, and $\gamma = 32.716°$. We denote these values by $\alpha_1, \beta_1, \gamma_1$ respectively. This leads to the following

Lemma 3.1. *All triangles with $\alpha \geq \beta \geq \gamma$ and $\sin\beta + \sin\gamma = 3/2$ satisfy $\gamma_0 \geq \gamma \geq \gamma_1$, and each such value of γ determines a unique such triangle in which $\beta_0 \leq \beta \leq \beta_1$ and $\alpha_0 \geq \alpha \geq \alpha_1$.*

Proof. First observe that $\gamma \geq 30°$ since if $\gamma < 30°$ then $\sin\gamma < 1/2$ and $\sin\beta > 1$. Also both $\gamma, \beta \leq 90°$ since only the largest angle α can be $> 90°$. It follows that $\sin\beta + \sin\gamma = 3/2$ determines a one-to-one relationship between β and γ and hence each such γ determines a unique triangle. Hence in the following paragraph we may consider α, β as functions of γ.

To show that the stated restrictions on α, β, γ are necessary we begin from the special case $\alpha_0, \beta_0, \gamma_0$ considered above. Since $\sin\beta + \sin\gamma = 3/2$ we have, differentiating with respect to γ, $\beta' \cos\beta + \gamma' \cos\gamma = 0$ and so as γ decreases, β increases. Also $\alpha' = -\beta' - \gamma' = \gamma' \left(\cos\gamma - \cos\beta\right) / \cos\beta$ which is negative since $\beta \geq \gamma$ and so as γ decreases, α decreases. This monotonic dependence continues until $\alpha = \beta$ and we arrive at the case $\alpha_1, \beta_1, \gamma_1$ as required.

We can now prove our main theorem.

Theorem 3.1. *Let $\alpha \geq \beta \geq \gamma$ be the angles of a triangle ABC. Then T is bounded if and only if $sin\beta + \sin\gamma > 3/2$.*

Proof. We prove that T is unbounded if $\sin\beta + \sin\gamma = 3/2$, the other possibilities having been dealt with in the previous section. The restrictions on α, β, γ from Lemma 3.1 will be introduced as and when required.

Let P be a point on BC at a distance $x\,(> 0)$ from C with P and B on opposite sides of C. Then for $PA+PB+PC$ we have $\sqrt{x^2 + b^2 + 2bx\cos\gamma} + (x+a) + x$ and for $PX + PY + PZ$ we have $0 + x\sin\gamma + (x+a)\sin\beta = 3x/2 + a\sin\beta$. Hence for x to be in T we require

$$\begin{aligned} \sqrt{x^2 + b^2 + 2bx\cos\gamma} + 2x + a &\geq 2\left(3x/2 + a\sin\beta\right), \text{ or} \\ \sqrt{x^2 + b^2 + 2bx\cos\gamma} &\geq x + a\left(2\sin\beta - 1\right) \end{aligned}$$

where both sides are positive since $\sin\beta \geq 1/2$. Hence we may square to obtain

$$\begin{aligned} x^2 + b^2 + 2bx\cos\gamma &\geq x^2 + 2ax\left(2\sin\beta - 1\right) + a^2\left(2\sin\beta - 1\right)^2 \\ 2x\left(b\cos\gamma - a\left(2\sin\beta - 1\right)\right) &\geq a^2\left(2\sin\beta - 1\right)^2 - b^2. \end{aligned}$$

Putting $a, b, c = 2R(\sin\alpha, \sin\beta, \sin\gamma)$, R being the circumradius of the triangle, and factorising, we get

$$\begin{aligned} &\frac{x}{R}\left(\sin\beta\cos\gamma - \sin\alpha(2\sin\beta - 1)\right) \\ \geq\ &\left(\sin\alpha(2\sin\beta - 1) - \sin\beta\right)\left(\sin\alpha(2\sin\beta - 1) + \sin\beta\right), \text{ or} \\ &\frac{x}{R}\left(\sin\alpha(2\sin\beta - 1) - \sin\beta\cos\gamma\right) \\ \leq\ &\left(\sin\beta - \sin\alpha(2\sin\beta - 1)\right)\left(\sin\beta + \sin\alpha(2\sin\beta - 1)\right) \end{aligned} \tag{3}$$

as the condition for x to be in T. On the right here we have two factors, the second of which is positive by inspection since $\sin\beta > 1/2$. The first may be written as

$$\sin\beta - \sin\alpha(2\sin\beta - 1) = (1 - (2\sin\beta - 1)(2\sin\alpha - 1))/2$$

which is positive since both $\sin\alpha, \sin\beta \in [1/2, 1)$. Writing $K := \sin\alpha(2\sin\beta - 1) - \sin\beta\cos\gamma$ and G for the right hand side of (3) we can write (3) as $xK/R \leq G$ where $G > 0$. Thus if $K \leq 0$ then (3) is satisfied for all $x \geq 0$, while if $K > 0$ it is satisfied for all x with $xK/R \leq G$, i.e. $x \in T$ for $x \leq RG/K$ but $x \in F$ for all $x > RG/K$. Thus we have to determine the sign of K. Here $K > 0$ if and only if

$$\begin{aligned} \sin\alpha(2\sin\beta - 1) &> \sin\beta\cos\gamma, \text{ or equivalently} \\ (\sin\beta\cos\gamma + \cos\beta\sin\gamma)(2\sin\beta - 1) &> \sin\beta\cos\gamma \text{ since } \sin\alpha = \sin(\beta+\gamma). \end{aligned}$$

On rearranging we get

$$\begin{aligned} \cos\beta\sin\gamma(2\sin\beta - 1) &> \sin\beta\cos\gamma(1 - 2\sin\beta + 1) \\ &= 2\sin\beta\cos\gamma(1 - \sin\beta). \end{aligned} \tag{4}$$

Before launching into the proof of (4), notice that both sides are equal when $\sin\beta = \sin\gamma = 3/4$ so there is some hope that a reasonable simplification can be found. In particular, if $\sin\beta = \sin\gamma = 3/4$ then $K = 0$ and the whole of the side BC produced in both directions lies in T.

Now square (4) (both sides are positive as previously found) and eliminate γ from $\sin\beta + \sin\gamma = 3/2$ to get

$$\left(1 - \sin^2\beta\right)(3/2 - \sin\beta)^2(2\sin\beta - 1)^2 > 4\sin^2\beta\left(1 - (3/2 - \sin\beta)^2\right)(1 - \sin\beta)^2$$

from which a factor of $(1 - \sin\beta)$ may be cancelled. With $s = \sin\beta$ this becomes

$$(s+1)\left(s^2 - 3s + 9/4\right)(2s-1)^2 > 4s^2\left(3s - s^2 - 5/4\right)(1-s)$$

which reduces to $8s^3 - 18s^2 + 21s - 9 = (4s - 3)\left(2s^2 - 3s + 3\right) > 0$. The quadratic term is positive-definite and we can deduce from this that $K > 0$ if and only if $4s - 3 > 0$, $\sin\beta > 3/4$, $\beta > \beta_0$. This shows that when $\beta > \gamma$ the extension of BC beyond C is first in T and eventually in F, while the extension of BC beyond B is always in T. Reversing the roles of β, γ completes the proof.

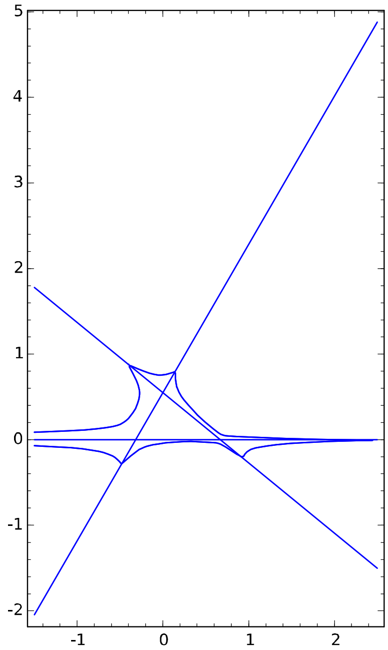

Figure 3

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7 REDGROVE PARK, CHELTENHAM, GL51 6QY, UK
E-mail address: `peter.walker@cantab.net`