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A NEW PROOF TO THE THEOREM OF COŞNIŢĂ

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Cezar Coşniţă (1910-1962) was a Romanian mathematician. In the international literature his name is known as Kosnitza, possible simplification because the accents used in Romanian. The statement of the theorem of Coşniţă is the following :

Theorem. *Let O be the circumcenter of triangle ABC and let Q_1, Q_2, O_3 be the circumcenters of triangles BOC, COA, respectively AOB. The lines AQ_1, BQ_2, CQ_3 are concurrent in a point P, called the Coşniţă' s point of triangle ABC.*

In [2] are given two geometric proofs to the above result. The point P and the center of the nine point circle of Euler O_9 are isogonal. Therefore the theorem of Coşniţă provides a geometric construction of the isogonal point of O_9.

The Coşniţă' s point of the triangle is the center $X(54)$ in the Kimberling's Encyclopedia of Triangle Centers and it is involved in many interesting properties.

The main purpose of this short note is to prove the theorem of Coşniţă by using complex numbers, a powerful technique in the study of many geometric properties (see for instance the references [2],[3],[4]). As a consequence of our proof, we obtain the complex coordinate of the Coşniţă's point, when the circumcenter of triangle ABC is the origin of the complex plane. In this respect we need the following two auxiliary results.

Lemma 1. Consider $A(a)$, $B(b)$, $C(c)$, $D(d)$ four points in the complex plane such that the lines AB and CD să intersects in the point $M(m)$, where m is the complex coordinates of M. Then

$$m = \frac{(a\bar{b} - \bar{a}b)(d-c) - (c\bar{d} - \bar{c}d)(b-a)}{(\bar{b} - \bar{a})(d-c) - (\bar{d} - \bar{c})(b-a)}$$

Proof. Because the points A, B and M are collinear we have

$$a\bar{b} - \bar{a}b - m(\bar{b} - \bar{a}) + \bar{m}(b-a) = 0. \tag{1}$$

The points C, D şi M are collinear, therefore

$$c\bar{d} - \bar{c}d - m(\bar{d} - \bar{c}) + \bar{m}(d-c) = 0. \tag{2}$$

Eliminating $\bar{m}$ from (1) and (2) we obtain the desired result. □

Lemma 2. Let ABC be a triangle and let Q be its circumcenter. Consider a, b, c, q be the complex coordinates of points A, B, C, Q. Then

$$q = \frac{|a|^2(b-c) + |b|^2(c-a) + |c|^2(a-b)}{\bar{a}(b-c) + \bar{b}(c-a) + \bar{c}(a-b)}.$$

Proof. We have $QA = QB = QC$. From the relation $QA = QB$ it follows $|q-a| = |q-b|$, that is

$$(q-a)(\bar{q}-\bar{a}) = (q-b)(\bar{q}-\bar{b}),$$

and we obtain

$$q(\bar{a}-\bar{b}) + \bar{q}(a-b) = |a|^2 - |b|^2. \tag{3}$$

Similarly, the relation $QA = QC$ is equivalent to $|q-a| = |q-b|$, hence

$$(q-a)(\bar{q}-\bar{a}) = (q-c)(\bar{q}-\bar{c}),$$

that is

$$q(\bar{a}-\bar{c}) + \bar{q}(a-c) = |a|^2 - |c|^2$$

By eliminating $\bar{q}$ from the above two equations we get the conclusion. □

Now we are in position to prove the Coşniţă theorem. Let $C(O; R)$ be the circumcircle of triangle ABC, where R is the circumradius. Without loss of generality we can assume that O is the origin of the complex plane. In this case we have $OA = OB = OC = R$, and the complex coordinate of the circumcenter O is 0. Denote by a, b, c, q_1, q_2, q_3 the complex coordinates of the points A, B, C, Q_1, Q_2, Q_3. From $|a| = |b| = |c| = R$, it follows $\overline{a}a = \overline{b}b = \overline{c}c = R^2$, hence we have $\overline{a} = \dfrac{R^2}{a}$, $\overline{b} = \dfrac{R^2}{b}$, $\overline{c} = \dfrac{R^2}{c}$.

Using the formula in Lemma 2, we obtain

$$q_1 = \frac{|b|^2 c + |c|^2(-b)}{\overline{b}c + \overline{c}(-b)} = \frac{R^2(c-b)}{R^2(\frac{c}{b} - \frac{b}{c})} = \frac{bc}{b+c}.$$

Therefore, we obtain

$$q_1 = \frac{bc}{b+c}, \overline{q}_1 = \frac{R^2}{b+c}.$$

Similarly, we get

$$q_2 = \frac{ac}{a+c}, \overline{q}_2 = \frac{R^2}{a+c}$$

and

$$q_3 = \frac{ab}{a+b}, \overline{q}_3 = \frac{R^2}{a+b}.$$

Let M be the intersection point of the lines AQ_1 and BQ_2 and let m be its complex coordinate. Applying the formula in Lemma 1, we have

$$m = \frac{(a\bar{q_1} - \bar{a}q_1)(q_2 - b) - (b\bar{q_2} - \bar{b}q_2)(q_1 - a)}{(\bar{q_1} - \bar{a})(q_2 - b) - (\bar{q_2} - \bar{b})(q_1 - a)}$$

From this relation easily follows

$$m = \frac{a^2b^2 + a^2c^2 + b^2c^2}{(a+b+c)(ab+ac+bc) - 4abc} \tag{4}$$

Consider N the intersection point of the lines AQ_1 and CQ_3 and n its complex coordinate. From the formula in Lemma 1, we obtain

$$n = \frac{(a\bar{q_1} - \bar{a}q_1)(q_3 - c) - (c\bar{q_3} - \bar{c}q_3)(q_1 - a)}{(\bar{q_1} - \bar{a})(q_3 - c) - (\bar{q_3} - \bar{c})(q_1 - a)},$$

therefore, after easy computations, it follows that n is also given by (4). In analogous way, if P is the intersection point of the lines BQ_2 and CQ_3 and p is its complex coordinate, we obtain the same expression for p as in (4). Finally, we obtain $m = n = p$, that is $M = N = P$.

Note that the complex coordinate of the Coşniţă's point, when the circumcenter of triangle ABC is the origin of the complex plane, has a nice symmetric form, that is

$$p = \frac{a^2b^2 + a^2c^2 + b^2c^2}{(a + b + c)(ab + ac + bc) - 4abc}.$$

References

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