INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 5 (2016), No. 2, 39 - 50

BIHARMONIC HYPERSURFACES IN $\mathbb{E}^6$ WITH CONSTANT SCALAR CURVATURE

RAM SHANKAR GUPTA

Abstract. In this paper, we study biharmonic hypersurfaces in $\mathbb{E}^6$ with constant scalar curvature. We prove that every such biharmonic hypersurface in Euclidean space $\mathbb{E}^6$ with at most four distinct principal curvatures must be minimal.

1. Introduction

The study of biharmonic submanifolds in Euclidean spaces was initiated by B. Y. Chen in mid 1980s. In particular, he proved that biharmonic surfaces in Euclidean 3-spaces are minimal. Based on the results of Dimitric in [10, 11], Chen [2] posed the following well-known conjecture in 1991:

The only biharmonic submanifolds of Euclidean spaces are the minimal ones.

The conjecture was later proved for hypersurfaces in Euclidean 4-spaces by Hasanis and Vlachos [15] and also for hypersurfaces with three distinct principal curvatures in $\mathbb{E}^5$ by Fu [16]. It was proved that the Chen's conjecture is true for hypersurfaces with three distinct principal curvatures in Euclidean space of arbitrary dimension and also for $\delta(2)$-ideal and $\delta(3)$-ideal hypersurfaces of a Euclidean space of arbitrary dimension [13, 7]. Also, it was proved that every biharmonic hypersurface with zero scalar curvature in $\mathbb{E}^5$ must be minimal [8]. Recently, it was proved that every biharmonic hypersurface in $\mathbb{E}^5$ must be minimal [14]. For more results on this topic see [3].

Chen's conjecture is not always true for submanifolds of semi-Euclidean spaces (see [4, 5, 6]). However, for hypersurfaces in semi-Euclidean spaces, Chen's conjecture is also right (see [5, 6, 9]). A. Arvanitoyeorgos et al. [1] proved that biharmonic Lorentzian hypersurfaces in Minkowski 4-spaces are minimal. It was proved that biharmonic hypersurface in semi-Euclidean 5-spaces with three distinct principal curvatures must be minimal [12].

In this paper, we study biharmonic hypersurfaces in $\mathbb{E}^6$ with constant

Keywords and phrases: Biharmonic hypersurface, Mean curvature vector.
(2010)Mathematics Subject Classification: 53D12, 53C40, 53C42
Received: 11.06.2015. In revised form: 25.05.2016. Accepted: 10.07.2016.

scalar curvature and prove that:

Theorem 1.1. *Every biharmonic hypersurface of constant scalar curvature in the Euclidean space $\mathbb{E}^6$ with at most four distinct principal curvatures is minimal.*

2. Preliminaries

Let (M, g) be a hypersurface isometrically immersed in a 6-dimensional Euclidean space $(\mathbb{E}^6, \overline{g})$ and $g = \overline{g}_{|M}$.

Let $\overline{\nabla}$ and ∇ denote linear connections on $\mathbb{E}^6$ and M, respectively. Then, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \ \forall \ X, Y \in \Gamma(TM), \tag{1}$$

$$\overline{\nabla}_X \xi = -A_\xi X, \tag{2}$$

where ξ is the unit normal vector to M, h is the second fundamental form and A is the shape operator. It is well known that the second fundamental form h and shape operator A are related by

$$\overline{g}(h(X,Y), \xi) = g(A_\xi X, Y). \tag{3}$$

The mean curvature is given by

$$H = \frac{1}{5}\text{trace}A. \tag{4}$$

The Gauss and Codazzi equations are given by

$$R(X,Y)Z = g(AY, Z)AX - g(AX, Z)AY, \tag{5}$$

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{6}$$

respectively, where R is the curvature tensor and

$$(\nabla_X A)Y = \nabla_X AY - A(\nabla_X Y) \tag{7}$$

for all $X, Y, Z \in \Gamma(TM)$.

A biharmonic submanifold in a Euclidean space is called proper biharmonic if it is not minimal. The necessary and sufficient conditions for M to be biharmonic in $\mathbb{E}^6$ is

$$\triangle H + H\,traceA^2 = 0, \tag{8}$$

$$A\,gradH + \frac{5}{2}H\,gradH = 0, \tag{9}$$

where H denotes the mean curvature. Also, the Laplace operator $\triangle$ of a scalar valued function f is given by [3]

$$\triangle f = -\sum_{i=1}^{5}(e_i e_i f - \nabla_{e_i} e_i f), \tag{10}$$

where $\{e_1, e_2, e_3, e_4, e_5\}$ is an orthonormal local tangent frame on M.

3. Biharmonic hypersurfaces in $\mathbb{E}^6$

In this section we study biharmonic hypersurfaces M in $\mathbb{E}^6$ with four distinct principal curvatures. We assume that the mean curvature is not constant and grad$H \neq 0$. Assuming non constant mean curvature implies the existence of an open connected subset U of M, with grad$_pH \neq 0$, for all $p \in U$. From (9), it is easy to see that grad$H$ is an eigenvector of the shape operator A with the corresponding principal curvature $-\frac{5}{2}H$. Without losing generality, we choose e_1 in the direction of grad$H$ and $\lambda_4 = \lambda_5 = \lambda$ as M has four distinct principal curvatures. Therefore shape operator A of hypersurfaces will take the following form with respect to a suitable frame $\{e_1, e_2, e_3, e_4, e_5\}$

$$A_H = \begin{pmatrix} -\frac{5}{2}H & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \lambda & \\ & & & & \lambda \end{pmatrix}. \tag{11}$$

The grad$H$ can be expressed as

$$gradH = \sum_{i=1}^{5} e_i(H)e_i. \tag{12}$$

As we have taken e_1 parallel to grad$H$, consequently

$$e_1(H) \neq 0, e_i(H) = 0, \qquad i = 2, \ldots, 5. \tag{13}$$

We express

$$\nabla_{e_i}e_j = \sum_{k=1}^{5} \omega_{ij}^k e_k, \qquad i, j = 1, \ldots, 5. \tag{14}$$

Using (14) and the compatibility conditions $(\nabla_{e_k}g)(e_i, e_i) = 0$ and $(\nabla_{e_k}g)(e_i, e_j) = 0$, we obtain

$$\omega_{ki}^i = 0, \qquad \omega_{ki}^j + \omega_{kj}^i = 0, \tag{15}$$

for $i \neq j$, and $i, j, k = 1, \ldots, 5$.

Taking $X = e_i, Y = e_j$ in (7) and using (11), (14), we get

$$(\nabla_{e_i}A)e_j = e_i(\lambda_j)e_j + \sum_{k=1}^{5} \omega_{ij}^k e_k(\lambda_j - \lambda_k).$$

Putting the value of $(\nabla_{e_i}A)e_j$ in (6), we find

$$e_i(\lambda_j)e_j + \sum_{k=1}^{5} \omega_{ij}^k e_k(\lambda_j - \lambda_k) = e_j(\lambda_i)e_i + \sum_{k=1}^{5} \omega_{ji}^k e_k(\lambda_i - \lambda_k),$$

whereby for $i \neq j = k$ and $i \neq j \neq k$, we obtain

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j = (\lambda_j - \lambda_i)\omega_{jj}^i, \tag{16}$$

$$(\lambda_i-\lambda_j)\omega_{ki}^j=(\lambda_k-\lambda_j)\omega_{ik}^j, \tag{17}$$

respectively, for distinct $i,j,k=1,...,5$.

Using (13), (14) and the fact that $[e_i\, e_j](H)=0=\nabla_{e_i}e_j(H)-\nabla_{e_j}e_i(H)=\omega_{ij}^1e_1(H)-\omega_{ji}^1e_1(H)$, for $i\neq j$ and $i,j=2,...,5$, we find

$$\omega_{ij}^1=\omega_{ji}^1. \tag{18}$$

From (4), we obtain that

$$\lambda_2+\lambda_3+2\lambda=\frac{15}{2}H,\quad \lambda\neq-\frac{5}{2}H. \tag{19}$$

Putting $i,j=4,5$, and $i\neq j$ in (16), we get

$$e_j(\lambda)=0,\qquad \text{for}\quad j=4,5. \tag{20}$$

Putting $i\neq1, j=1$ in (16) and using (13) and (15), we find

$$\omega_{1i}^1=0,\qquad i=1,...,5. \tag{21}$$

Putting $i=1, j=2,...,5$, in (16) and using (15), we have

$$\omega_{22}^1=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1},\quad \omega_{33}^1=\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1},\quad \omega_{jj}^1=\frac{e_1(\lambda)}{\lambda-\lambda_1},\qquad j=4,5. \tag{22}$$

Putting $i=2,3, j=4,5$, in (16), we find

$$\omega_{jj}^2=\frac{e_2(\lambda)}{\lambda-\lambda_2},\quad \omega_{jj}^3=\frac{e_3(\lambda)}{\lambda-\lambda_3},\qquad j=4,5. \tag{23}$$

Putting $i=1, j\neq k$, and $j,k=4,5$, in (17), we obtain

$$\omega_{k1}^j=0,\qquad j\neq k,\quad\text{and}\quad j,k=4,5. \tag{24}$$

Putting $i=2,3, j\neq k$, and $j,k=4,5$, in (17), we have

$$\omega_{k2}^j=0,\quad \omega_{k3}^j=0,\qquad j\neq k,\quad \text{and}\quad j,k=4,5. \tag{25}$$

Putting $i=2,3, j=1$, and $k=4,5$, in (17), and using (18) we get

$$\omega_{k2}^1=\omega_{2k}^1=\omega_{k3}^1=\omega_{3k}^1=0,\qquad k=4,5. \tag{26}$$

Putting $i=1, j=2,3$ and $k=4,5$, in (17), and using (18) we find

$$\omega_{1k}^2=\omega_{k1}^2=\omega_{1k}^3=\omega_{k1}^3=0,\qquad k=4,5. \tag{27}$$

Now, using (14) and (20)$\sim$(27), we have:

Lemma 3.1. *Let M be a biharmonic hypersurface of non-constant mean curvature with four distinct principal curvatures in Euclidean space $\mathbb{E}^6$, having the shape operator given by (11) with respect to a suitable orthonormal frame $\{e_1,e_2,e_3,e_4,e_5\}$. Then,*

$$\nabla_{e_1}e_1=\nabla_{e_1}e_2=\nabla_{e_1}e_3=0, \nabla_{e_1}e_4=\omega_{14}^5e_5, \nabla_{e_1}e_5=\omega_{15}^4e_4,\ \nabla_{e_i}e_1=-\omega_{ii}^1e_i,\ i=2,\ldots,5,$$

$$\nabla_{e_i}e_i=\sum_{i\neq l,l=1}^{5}\omega_{ii}^le_l,\ i=2,\ldots,5,\ \nabla_{e_i}e_j=\sum_{i\neq j,k=2}^{5}\omega_{ij}^ke_k,\ i=2,3,\quad \textit{and}\quad j=2,\ldots,5,$$

$$\nabla_{e_4}e_2=\omega_{42}^3e_3+\omega_{42}^4e_4, \nabla_{e_4}e_3=\omega_{43}^2e_2+\omega_{43}^4e_4, \nabla_{e_4}e_5=\omega_{45}^4e_4,$$

$$\nabla_{e_5}e_2=\omega_{52}^3e_3+\omega_{52}^5e_5, \nabla_{e_5}e_3=\omega_{53}^2e_2+\omega_{53}^5e_5, \nabla_{e_5}e_4=\omega_{54}^5e_5,$$

where ω_{ij}^i satisfy (15) and (16).

Next, we have

Lemma 3.2. *Let M be a biharmonic hypersurface of non-constant mean curvature with four distinct principal curvatures in Euclidean space $\mathbb{E}^6$, having the shape operator given by (11) with respect to a suitable orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$. Then, $g(R(e_1, e_i)e_1, e_i)$, $g(R(e_1, e_i)e_i, e_j)$ and $g(R(e_i, e_j)e_i, e_1)$ give the following:*

$$e_1(\omega_{ii}^1) - (\omega_{ii}^1)^2 = \lambda_1\lambda_i, \quad i = 2, \ldots, 5. \tag{28}$$

$$e_1(\omega_{ii}^j) - \omega_{ii}^j\omega_{ii}^1 = 0, \quad i \neq j, \ j = 2, 3 \quad \textit{and} \quad i = 2, \ldots, 5, \tag{29}$$

and

$$e_j(\omega_{ii}^1) + \omega_{ii}^j\omega_{jj}^1 - \omega_{ii}^j\omega_{ii}^1 = 0, \ i \neq j, \ j = 2, 3 \quad \textit{and} \quad i = 2, \ldots, 5, \tag{30}$$

respectively.

Proof. Using (5), (11) and Lemma 3.1, we have

$$g(R(e_1, e_i)e_1, e_i) = g(Ae_i, e_1)g(Ae_1, e_i) - g(Ae_1, e_1)g(Ae_i, e_i) = -\lambda_1\lambda_i, \tag{31}$$

or,

$$\begin{aligned} -\lambda_1\lambda_i &= g(\nabla_{e_1}\nabla_{e_i}e_1 - \nabla_{e_i}\nabla_{e_1}e_1 - \nabla_{[e_1 e_i]}e_1, e_i) \\ &= g(\nabla_{e_1}(-\omega_{ii}^1 e_i) - \omega_{ii}^1\nabla_{e_i}e_1, e_i) \\ &= g(-e_1(\omega_{ii}^1)e_i + (\omega_{ii}^1)^2 e_i, e_i) = -e_1(\omega_{ii}^1) + (\omega_{ii}^1)^2, \end{aligned} \tag{32}$$

for $i = 2, 3$.

Also, using (5), (11) and Lemma 3.1, we have

$$g(R(e_1, e_4)e_1, e_4) = g(Ae_4, e_1)g(Ae_1, e_4) - g(Ae_1, e_1)g(Ae_4, e_4) = -\lambda_1\lambda_4, \tag{33}$$

or,

$$\begin{aligned} -\lambda_1\lambda_4 &= g(\nabla_{e_1}\nabla_{e_4}e_1 - \nabla_{e_4}\nabla_{e_1}e_1 - \nabla_{[e_1 e_4]}e_1, e_4) \\ &= g(\nabla_{e_1}(-\omega_{44}^1 e_4) - \omega_{14}^5\nabla_{e_5}e_1 - \omega_{44}^1\nabla_{e_4}e_1, e_4) = g(-e_1(\omega_{44}^1)e_4 + (\omega_{44}^1)^2 e_4, e_4) \\ &= -e_1(\omega_{44}^1) + (\omega_{44}^1)^2. \end{aligned} \tag{34}$$

Similarly, $g(R(e_1, e_5)e_1, e_5)$ gives

$$-\lambda_1\lambda_5 = -e_1(\omega_{55}^1) + (\omega_{55}^1)^2. \tag{35}$$

Combining (32), (34) and (35), we get (28).

Now, using (5), (11) and Lemma 3.1, we have

$$g(R(e_1, e_i)e_i, e_j) = g(Ae_i, e_i)g(Ae_1, e_j) - g(Ae_1, e_i)g(Ae_i, e_j) = 0, \tag{36}$$

for $i \neq j$ and $i, j = 2, \ldots, 5$. Hence,

$$\begin{aligned} 0 &= g(\nabla_{e_1}\nabla_{e_i}e_i - \nabla_{e_i}\nabla_{e_1}e_i - \nabla_{[e_1 e_i]}e_i, e_j) = g(\nabla_{e_1}(\sum_{i\neq l, l=1}^{5}\omega_{ii}^l e_l) \\ &-\omega_{ii}^1\nabla_{e_i}e_i, e_j) = g((\sum_{i\neq l, l=1}^{5} e_1(\omega_{ii}^l)e_l) - \omega_{ii}^1(\sum_{i\neq l, l=1}^{5}\omega_{ii}^l e_l), e_j) = e_1(\omega_{ii}^j) - \omega_{ii}^1\omega_{ii}^j, \end{aligned} \tag{37}$$

for $i \neq j$ and $i, j = 2, 3$.

Evaluating $g(R(e_1,e_4)e_4,e_j)$, using (36) and Lemma 3.1, we find

$$\begin{aligned}(38)\quad 0 &= g(\nabla_{e_1}\nabla_{e_4}e_4-\nabla_{e_4}\nabla_{e_1}e_4-\nabla_{[e_1e_4]}e_4,e_j)=g(\nabla_{e_1}(\sum_{i\neq l,l=1}^{5}\omega_{44}^{l}e_l)\\ &-\nabla_{e_4}(\omega_{14}^{5}e_5)-\omega_{14}^{5}\nabla_{e_5}e_4-\omega_{44}^{1}\nabla_{e_4}e_4,e_j)=g((\sum_{i\neq l,l=1}^{5}e_1(\omega_{44}^{l})e_l)\\ &-\omega_{44}^{1}(\sum_{i\neq l,l=1}^{5}\omega_{44}^{l}e_l),e_j)=e_1(\omega_{44}^{j})-\omega_{44}^{1}\omega_{44}^{j},\end{aligned}$$

for $j=2,3$.

Similarly, evaluating $g(R(e_1,e_5)e_5,e_j)$ gives

$$(39)\qquad e_1(\omega_{55}^{j})-\omega_{55}^{1}\omega_{55}^{j}=0,$$

for $j=2,3$.

Combining (37), (38) and (39), we get (29).

Next, using (5), (11) and Lemma 3.1, we obtain

$$(40)\qquad g(R(e_i,e_j)e_i,e_1)=g(Ae_j,e_i)g(Ae_i,e_1)-g(Ae_i,e_i)g(Ae_j,e_1)=0,$$

for $i\neq j$, and $i,j=2,\dots,5$. Therefore

$$\begin{aligned}(41)\quad 0 &= g(\nabla_{e_i}\nabla_{e_j}e_i-\nabla_{e_j}\nabla_{e_i}e_i-\nabla_{[e_ie_j]}e_i,e_1)=g(\nabla_{e_i}(\sum_{j\neq i,k=2}^{5}\omega_{ji}^{k}e_k)\\ &-\nabla_{e_j}(\sum_{i\neq l,l=1}^{5}\omega_{ii}^{l}e_l)-\nabla_{\nabla_{e_i}e_j-\nabla_{e_j}e_i}e_i,e_1)=g(\sum_{j\neq i,k=2}^{5}(e_i(\omega_{ji}^{k})e_k+\omega_{ji}^{k}\nabla_{e_i}e_k),e_1)\\ &-g(\sum_{i\neq l,l=1}^{5}(e_j(\omega_{ii}^{l})e_l+\omega_{ii}^{l}\nabla_{e_j}e_l),e_1)-g(\nabla_{\nabla_{e_i}e_j-\nabla_{e_j}e_i}e_i,e_1),\end{aligned}$$

for $i\neq j$, and $i,j=2,3$.

Now,

$$g(\sum_{j\neq i,k=2}^{5}(e_i(\omega_{ji}^{k})e_k+\omega_{ji}^{k}\nabla_{e_i}e_k),e_1)=0,$$

as $\nabla_{e_i}e_k$ is not having component along e_1 for $i\neq k$ and for $i=k$, we have $\omega_{ji}^{i}=0$.

Similarly,

$$g(\sum_{i\neq l,l=1}^{5}(e_j(\omega_{ii}^{l})e_l+\omega_{ii}^{l}\nabla_{e_j}e_l),e_1)=e_j(\omega_{ii}^{1})+\omega_{jj}^{1}\omega_{ii}^{j}$$

and

$$g(\nabla_{\nabla_{e_i}e_j}e_i,e_1)=g(\sum_{k=2,i\neq j}^{5}\omega_{ij}^{k}\nabla_{e_k}e_i,e_1)=\omega_{ij}^{i}g(\nabla_{e_i}e_i,e_1)=-\omega_{ii}^{j}\omega_{ii}^{1},$$

$$g(\nabla_{\nabla_{e_j}e_i}e_i, e_1) = g(\sum_{k=2, i\neq j}^{5} \omega_{ji}^k \nabla_{e_k} e_i, e_1) = \omega_{ji}^i g(\nabla_{e_i} e_i, e_1) = 0.$$

Therefore, we get

$$e_j(\omega_{ii}^1) + \omega_{jj}^1 \omega_{ii}^j - \omega_{ii}^j \omega_{ii}^1 = 0, \tag{42}$$

for $i \neq j$, and $i, j = 2, 3$.

Evaluating $g(R(e_4, e_j)e_4, e_1)$ and $g(R(e_5, e_j)e_5, e_1)$ using (40) and Lemma 3.1, we obtain

$$e_j(\omega_{44}^1) + \omega_{jj}^1 \omega_{44}^j - \omega_{44}^j \omega_{44}^1 = 0, \tag{43}$$

and

$$e_j(\omega_{55}^1) + \omega_{jj}^1 \omega_{55}^j - \omega_{55}^j \omega_{55}^1 = 0, \tag{44}$$

for $j = 2, 3$.

Combining (42), (43) and (44), we get (30). Whereby proof of Lemma 3.2 is complete.

From (11), we find

$$\text{trace}A^2 = \tfrac{25H^2}{4} + \lambda_2^2 + \lambda_3^2 + 2\lambda^2.$$

Evaluating scalar curvature of the hypersurface, using (2.5) and (3.1), we get

$$\rho = \frac{75H^2}{4} - \lambda_2^2 - \lambda_3^2 - 2\lambda^2 = \frac{75H^2}{4} - \sum_{j=2}^{5} \lambda_j^2, \tag{45}$$

where ρ denotes the scalar curvature.

Using Lemma 3.1, (10), (13), (45) and putting the value of $\text{trace}A^2$ in (8), we obtain

$$-e_1 e_1(H) + \sum_{j=2}^{5} \omega_{jj}^1 e_1(H) + 25H^3 - \rho H = 0. \tag{46}$$

Using (13), Lemma 3.1, and the fact that $[e_i\ e_1](H) = 0 = \nabla_{e_i} e_1(H) - \nabla_{e_1} e_i(H)$, for $i = 2, \ldots, 5$, we find

$$e_i e_1(H) = 0. \tag{47}$$

Also, from (47) and using the fact that $[e_i\ e_1](e_1(H)) = 0 = \nabla_{e_i} e_1(e_1(H)) - \nabla_{e_1} e_i(e_1(H))$, we obtain

$$e_i e_1 e_1(H) = 0, \qquad i = 2, \ldots, 5. \tag{48}$$

Now, we have:

Lemma 3.3. *Let M be a biharmonic hypersurface with four distinct principal curvatures in Euclidean space $\mathbb{E}^6$, having the shape operator given by (3.1) with respect to a suitable orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$. If the scalar curvature is constant, then*

$$\omega_{22}^4 = \omega_{33}^4 = \omega_{22}^5 = \omega_{33}^5 = 0.$$

Proof. The equation (19) can be written as

$$\sum_{j=2}^{5} \lambda_j = \frac{15H}{2}. \tag{49}$$

Differentiating (49) with respect to e_i and using (13) and (20), we get

$$e_i(\lambda_2) + e_i(\lambda_3) = 0, \quad i = 4, 5. \tag{50}$$

Differentiating (45) with respect to e_i and using (50) and (20), we find

$$(\lambda_3 - \lambda_2) e_i(\lambda_3) = 0, \tag{51}$$

for $i = 4, 5$. From (51) and (50), we get $e_i(\lambda_3) = e_i(\lambda_2) = 0$. Which by use of (16) completes the proof of the Lemma.

Next, we have:

Lemma 3.4. *Let M be a biharmonic hypersurface with four distinct principal curvatures in Euclidean space $\mathbb{E}^6$, having the shape operator given by (11) with respect to a suitable orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$. If the scalar curvature is constant, then*

$$e_i(\omega_{ii}^1) = -\sum_{i \neq j, j=2}^{5} \omega_{jj}^i [\omega_{jj}^1 - \omega_{ii}^1], \tag{52}$$

$$e_i(\omega_{ii}^1) = -\frac{1}{(\lambda_i - \lambda_1)} \sum_{i \neq j, j=2}^{5} \omega_{jj}^i (\omega_{jj}^1 - \omega_{ii}^1)(2\lambda_j - \lambda_i - \lambda_1), \tag{53}$$

$$\sum_{j \neq i, j=2}^{5} \omega_{jj}^i [(\omega_{jj}^1 - \omega_{ii}^1)(\lambda_j - \lambda_i)] = 0, \tag{54}$$

$$\sum_{j \neq i, j=2}^{5} \omega_{jj}^i (\lambda_j - \lambda_i)^2 = 0, \tag{55}$$

$$\sum_{j \neq i, j=2}^{5} \omega_{jj}^i (\lambda_j - \lambda_i)[(\omega_{jj}^1 (3\lambda_j \lambda_i - 2\lambda_1) - 2\omega_{ii}^1)(\lambda_i - \lambda_1)] = 0, \tag{56}$$

for $i = 2, 3$.

Proof. Differentiating (46) with respect to e_i and using (13), (47) and (48), we find

$$\sum_{j=2}^{5} e_i(\omega_{jj}^1) e_1(H) = 0,$$

or,

$$e_i(\omega_{ii}^1) + \sum_{i \neq j, j=2}^{5} e_i(\omega_{jj}^1) = 0, \tag{57}$$

whereby using (30), we find (52).

Differentiating (49) with respect to e_i and using (13), we get

$$e_i(\lambda_i) + \sum_{j\neq i, j=2}^{5} e_i(\lambda_j) = 0, \quad i = 2, 3. \tag{58}$$

Next, differentiating (58) with e_1, we have

$$e_1 e_i(\lambda_i) + \sum_{i\neq j, j=2}^{5} e_1 e_i(\lambda_j) = 0. \tag{59}$$

From Lemma 3.1, we get $e_i e_1 - e_1 e_i = \nabla_{e_i} e_1 - \nabla_{e_1} e_i = -\omega_{ii}^1 e_i$, for $i = 2, 3$. Therefore, equation (59) can be written as

$$e_i e_1(\lambda_i) + \omega_{ii}^1 e_i(\lambda_i) + \sum_{i\neq j, j=2}^{5} e_1 e_i(\lambda_j) = 0. \tag{60}$$

Using (16) and (58) in (60), we get

$$\begin{aligned} e_i(\omega_{ii}^1)(\lambda_i - \lambda_1) + \omega_{ii}^1 e_i(\lambda_i) - \sum_{i\neq j, j=2}^{5} \omega_{ii}^1 e_i(\lambda_j) \\ + \sum_{i\neq j, j=2}^{5} [e_1(\omega_{jj}^i)(\lambda_j - \lambda_i) + \omega_{jj}^i e_1(\lambda_j - \lambda_i)] = 0. \end{aligned} \tag{61}$$

Using (16), (58) and (29) in (61), we find

$$\begin{aligned} e_i(\omega_{ii}^1)(\lambda_i - \lambda_1) - 2\sum_{i\neq j, j=2}^{5} \omega_{ii}^1 e_i(\lambda_j) + \sum_{i\neq j, j=2}^{5} [\omega_{jj}^1 \omega_{jj}^i (\lambda_j - \lambda_i) \\ + \omega_{jj}^i(\omega_{jj}^1(\lambda_j - \lambda_1) - \omega_{ii}^1(\lambda_i - \lambda_1))] = 0. \end{aligned} \tag{62}$$

Using (16) in (62), we obtain (53).
Eliminating $e_i(\omega_{ii}^1)$ from (52) and (53), we obtain (54).

Next, differentiating (45) with respect to e_i, we get

$$\sum_{j=2}^{5} \lambda_j e_i(\lambda_j) = 0, \tag{63}$$

or,

$$\lambda_i e_i(\lambda_i) + \sum_{j\neq i, j=2}^{5} \lambda_j e_i(\lambda_j) = 0, \quad i = 2, 3. \tag{64}$$

Using (58) and (16) in (64), we obtain (55).

Now, differentiating (55) with respect to e_1, we get

$$\sum_{j\neq i, j=2}^{5} e_1(\omega_{jj}^i)(\lambda_j - \lambda_i)^2 + 2\omega_{jj}^i(\lambda_j - \lambda_i) e_1(\lambda_j - \lambda_i) = 0. \tag{65}$$

Using (29) and (16) in (65), we find (56). Which completes the proof of the Lemma.

Further, we have

Lemma 3.5. *Let M be a biharmonic hypersurface with four distinct principal curvatures in Euclidean space $\mathbb{E}^6$, having the shape operator given by (11) with respect to a suitable orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$. If the scalar curvature is constant, then*

$$\omega_{33}^2 = \omega_{44}^2 = \omega_{55}^2 = 0, \quad \omega_{22}^3 = \omega_{44}^3 = \omega_{55}^3 = 0.$$

Proof. Putting $i = 2, 3$ in (54), we find following:

$$\omega_{33}^2[(\omega_{33}^1 - \omega_{22}^1)(\lambda_3 - \lambda_2)] + 2\omega_{44}^2[(\omega_{44}^1 - \omega_{22}^1)(\lambda - \lambda_2)] = 0, \tag{66}$$

and

$$\omega_{22}^3[(\omega_{33}^1 - \omega_{22}^1)(\lambda_3 - \lambda_2)] + 2\omega_{44}^3[(\omega_{44}^1 - \omega_{33}^1)(\lambda - \lambda_3)] = 0, \tag{67}$$

respectively.

Similarly, by putting $i = 2, 3$ in (55), we get following:

$$\omega_{33}^2(\lambda_3 - \lambda_2)^2 + 2\omega_{44}^2(\lambda - \lambda_2)^2 = 0, \tag{68}$$

and

$$\omega_{22}^3(\lambda_3 - \lambda_2)^2 + 2\omega_{44}^3(\lambda - \lambda_3)^2 = 0, \tag{69}$$

respectively.

Similarly, by putting $i = 2, 3$ in (56), we get following:

$$\begin{aligned} &\omega_{33}^2(\lambda_3 - \lambda_2)[\omega_{33}^1(3\lambda_3 - \lambda_2 - 2\lambda_1) - 2\omega_{22}^1(\lambda_2 - \lambda_1)] \\ &+2\omega_{44}^2(\lambda - \lambda_2)[\omega_{44}^1(3\lambda - \lambda_2 - 2\lambda_1) - 2\omega_{22}^1(\lambda_2 - \lambda_1)] = 0, \end{aligned} \tag{70}$$

and

$$\begin{aligned} &\omega_{22}^3(\lambda_2 - \lambda_3)[\omega_{22}^1(3\lambda_2 - \lambda_3 - 2\lambda_1) - 2\omega_{33}^1(\lambda_3 - \lambda_1)] \\ &+2\omega_{44}^3(\lambda - \lambda_3)[\omega_{44}^1(3\lambda - \lambda_3 - 2\lambda_1) - 2\omega_{33}^1(\lambda_3 - \lambda_1)] = 0, \end{aligned} \tag{71}$$

respectively.

We claim that ω_{33}^2=0 and ω_{44}^2=0. In fact, if $\omega_{33}^2 \neq 0$ and $\omega_{44}^2 \neq 0$, then the value of determinant formed by coefficients of ω_{33}^2 and ω_{44}^2 in (66) and (68) and the value of determinant formed by coefficients of ω_{33}^2 and ω_{44}^2 in (68) and (70) will be zero. Therefore, we obtain that

$$(\lambda - \lambda_2)(\omega_{33}^1 - \omega_{22}^1) - (\omega_{44}^1 - \omega_{22}^1)(\lambda_3 - \lambda_2) = 0, \tag{72}$$

and

$$\begin{aligned} &(\lambda - \lambda_2)[\omega_{33}^1(3\lambda_3 - \lambda_2 - 2\lambda_1) - 2\omega_{22}^1(\lambda_2 - \lambda_1)] \\ &-(\lambda_3 - \lambda_2)[\omega_{44}^1(3\lambda - \lambda_2 - 2\lambda_1) - 2\omega_{22}^1(\lambda_2 - \lambda_1)] = 0, \end{aligned} \tag{73}$$

respectively.

Eliminating ω_{33}^1, from (72) and (73), we get

$$\omega_{22}^1 = \omega_{44}^1, \tag{74}$$

which is not possible as from (28), it gives $\lambda_2 = \lambda_4$, a contradiction. Therefore, $\omega_{33}^2 = 0$ and $\omega_{44}^2 = 0$.

In an analogous manner, using (67), (69) and (71), we find that $\omega_{22}^3 = 0$ and $\omega_{44}^3 = 0$. From (23), we have $\omega_{44}^2 = \omega_{55}^2$ and $\omega_{44}^3 = \omega_{55}^3$. Therefore, $\omega_{55}^2 = \omega_{55}^3 = 0$, which completes the proof of the Lemma.

Now, we have:

Lemma 3.6. *Let M be a biharmonic hypersurface with four distinct principal curvatures in Euclidean space $\mathbb{E}^6$, having the shape operator given by (11) with respect to a suitable orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$. If the scalar curvature is constant, then $\omega_{23}^4 = \omega_{32}^4 = \omega_{42}^3 = \omega_{24}^3 = \omega_{34}^2 = \omega_{43}^2 = 0$, and $\omega_{23}^5 = \omega_{32}^5 = \omega_{52}^3 = \omega_{25}^3 = \omega_{35}^2 = \omega_{53}^2 = 0$.*

Proof. From Lemma 3.5 and Lemma 3.1, we have

$$\nabla_{e_2} e_3 = \omega_{23}^4 e_4 + \omega_{23}^5 e_5, \quad \nabla_{e_3} e_2 = \omega_{32}^4 e_4 + \omega_{32}^5 e_5. \tag{75}$$

Now, evaluating $g(R(e_1, e_3)e_2, e_4)$, $g(R(e_1, e_2)e_3, e_4)$, $g(R(e_1, e_3)e_2, e_5)$ and
$g(R(e_1, e_2)e_3, e_5)$, using (5), (11) and (75), we find

$$e_1(\omega_{32}^4) + \omega_{32}^5 \omega_{15}^4 - \omega_{32}^4 \omega_{33}^1 = 0, \tag{76}$$

$$e_1(\omega_{23}^4) + \omega_{23}^5 \omega_{15}^4 - \omega_{23}^4 \omega_{22}^1 = 0, \tag{77}$$

$$e_1(\omega_{32}^5) + \omega_{32}^4 \omega_{14}^5 - \omega_{32}^5 \omega_{33}^1 = 0, \tag{78}$$

and

$$e_1(\omega_{23}^5) + \omega_{23}^4 \omega_{14}^5 - \omega_{23}^5 \omega_{22}^1 = 0, \tag{79}$$

respectively.

Putting $j = 4, k = 2, i = 3$ in (17), we get

$$(\lambda_2 - \lambda)\omega_{32}^4 = (\lambda_3 - \lambda)\omega_{23}^4. \tag{80}$$

Putting $j = 5, k = 2, i = 3$ in (17), we get

$$(\lambda_2 - \lambda)\omega_{32}^5 = (\lambda_3 - \lambda)\omega_{23}^5. \tag{81}$$

Differentiating (80) with e_1, and using (76) and (77), we find

$$\begin{aligned}(e_1(\lambda_2) - e_1(\lambda))\omega_{32}^4 &+ (\lambda_2 - \lambda)(\omega_{32}^4 \omega_{33}^1 - \omega_{32}^5 \omega_{15}^4) \\ &= (e_1(\lambda_3) - e_1(\lambda))\omega_{23}^4 + (\lambda_3 - \lambda)(\omega_{23}^4 \omega_{22}^1 - \omega_{23}^5 \omega_{15}^4).\end{aligned} \tag{82}$$

Putting the values of $e_1(\lambda_2), e_1(\lambda_3)$ and $e_1(\lambda)$ from (16) and using (81) in (82), we obtain

$$\omega_{32}^4(\omega_{22}^1 - \omega_{44}^1) = \omega_{23}^4(\omega_{33}^1 - \omega_{44}^1). \tag{83}$$

Substituting the value of ω_{23}^4 from (80) in (83), we find

$$\omega_{32}^4[(\lambda - \lambda_2)(\omega_{33}^1 - \omega_{22}^1) - (\omega_{44}^1 - \omega_{22}^1)(\lambda_3 - \lambda_2)] = 0. \tag{84}$$

As from (72), we have seen that assuming
$(\lambda - \lambda_2)(\omega_{33}^1 - \omega_{22}^1) - (\omega_{44}^1 - \omega_{22}^1)(\lambda_3 - \lambda_2) = 0$,
leads to contradiction, therefore from (84), we obtain $\omega_{32}^4 = 0$, which together with (80) gives $\omega_{23}^4 = 0$. Also, from (15), we get $\omega_{34}^2 = -\omega_{32}^4$ and $\omega_{23}^4 = -\omega_{24}^3$. Consequently, we obtain $\omega_{34}^2 = 0$, and $\omega_{24}^3 = 0$, which together with (17) gives $\omega_{43}^2 = 0$ and $\omega_{42}^3 = 0$.

Similarly, using (81), (78) and (79), we can show that

$$\omega_{23}^5 = \omega_{32}^5 = \omega_{52}^3 = \omega_{25}^3 = \omega_{35}^2 = \omega_{53}^2 = 0,$$

whereby proof of Lemma 3.6 is complete.

4. Proof of the theorem

Evaluating $g(R(e_2, e_3)e_2, e_3)$, $g(R(e_2, e_4)e_2, e_4)$, and $g(R(e_3, e_4)e_3, e_4)$ and keeping in view Lemma 3.1, Lemma 3.5 and Lemma 3.6, we find

$$-\omega_{22}^1\omega_{33}^1 = \lambda_2\lambda_3, \quad -\omega_{22}^1\omega_{44}^1 = \lambda_2\lambda, \quad -\omega_{33}^1\omega_{44}^1 = \lambda_3\lambda. \tag{85}$$

From (85), we get

$$(\omega_{22}^1)^2 + (\lambda_2)^2 = 0, \quad (\omega_{33}^1)^2 + (\lambda_3)^2 = 0, \quad (\omega_{44}^1)^2 + \lambda^2 = 0. \tag{86}$$

From (86), we find $\lambda_2 = \lambda_3 = \lambda = 0$. Which is a contradiction of four distinct principal curvatures.

Also, the cases of three or two distinct principal curvatures for biharmonic hypersurfaces in Euclidean space of arbitrary dimension has already been proved in [11, 13] and have also concluded that H must be zero, whereby showing that the proof of Theorem 1.1 is complete.

References

[1] Arvanitoyeorgos, A., Defever, F., Kaimakamis, G., and Papantoniou, V., *Biharmonic Lorentzian hypersurfaces in* $\mathbb{E}_1^4$, Pac. J. Math., **229(2)(2007)** 293–305.

[2] Chen, B. Y., *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math., **17(2)(1991)** 169–188.

[3] Chen, B. Y., *Total mean curvature and submanifolds of finite type*, 2nd edition, World Scientific, Hackensack, NJ, 2015.

[4] Chen, B. Y., *Classification of marginally trapped Lorentzian flat surfaces in* $\mathbb{E}_1^4$ *and its application to biharmonic surfaces*, J. Math. Anal. Appl.,**340 (2008)** 861–875.

[5] Chen, B. Y., Ishikawa, S., *Biharmonic surfaces in pseudo-Euclidean spaces*, Mem. Fac. Sci. Kyushu Univ., **A 45(1991)** 323–347.

[6] Chen, B. Y., Ishikawa, S., *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*, Kyushu J. Math., **52 (1998)** 1–18.

[7] Chen, B. Y., Munteanu, M. I.,*Biharmonic ideal hypersurfaces in Euclidean spaces*, Differ. Geom. Appl., **31 (2013)** 1–16.

[8] Deepika, Gupta, R. S., *Biharmonic hypersurfaces in* $\mathbb{E}^5$ *with zero scalar curvature*, Afr. Diaspora J. Math., **18(1)(2015)** 12–26.

[9] Defever, F., Kaimakamis, G., Papantoniou, V., *Biharmonic hypersurfaces of the 4-dimensional semi-Euclidean space* $\mathbb{E}_s^4$, J. Math. Anal. Appl., **315 (2006)** 276–286.

[10] Dimitri´c, I., *Quadric representation and submanifolds of finite type*, Doctoral thesis, Michigan State University (1989).

[11] Dimitri´c, I., *Submanifolds of* $\mathbb{E}^n$ *with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sinca, **20 (1992)** 53–65.

[12] Gupta, R. S., *Biharmonic hypersurfaces in* $\mathbb{E}_s^5$, An. St. Univ. Al. I. Cuza, (accepted).

[13] Gupta, R. S., *On biharmonic hypersurfaces in Euclidean space of arbitrary dimension*, Glasgow Math. J., **57 (2015)** 633–642.

[14] Gupta, R. S., Sharfuddin, A., *Biharmonic hypersurfaces in Euclidean space* $\mathbb{E}^5$, J. Geom., (2016), DOI:10.1007/s00022-015-0310-2.

[15] Hasanis, T., Vlachos, T., *Hypersurfaces in* $\mathbb{E}^4$ *with harmonic mean curvature vector field*, Math. Nachr., **172 (1995)** 145–169.

[16] Fu, Y., *Biharmonic hypersurfaces with three distinct principal curvatures in the Euclidean 5-space*, Journal of Geometry and Physics, **75 (2014)** 113–119.

UNIVERSITY SCHOOL OF BASIC AND APPLIED SCIENCES
GURU GOBIND SINGH INDRAPRASTHA UNIVERSITY
SECTOR-16C, DWARKA NEW DELHI-110078 INDIA
E-mail address: ramshankar.gupta@gmail.com