INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 5 (2016), No. 1, 19 - 30

Equal Area Triangles Inscribed in a Triangle

SÁNDOR NAGYDOBAI KISS

Abstract. We examine the condition that two or more inscribed triangles in a triangle have same area, and the relations between the equal area triangles and the isotomic points.

1. Introduction

A triangle XYZ is said to be inscribed in a triangle ABC if X lies on BC, Y lies on CA, and Z lies on AB. In mathematics literature there are not many references about the inscribed triangles in a triangle. Personally I have found two documents [1], [2] in which equilateral triangles inscribed in a triangle are discussed.

In this paper, in section 1 we examine the condition that two or more inscribed triangles have the same area and in section 2 we enounce a theorem, which permits the construction of inscribed equal area triangles in a triangle. In section 3 we examine the relations between the equal area triangles and the isotomic points. Finally we give some properties of the inscribed equal area triangles.

We use the barycentric coordinates with respect to the triangle ABC: $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. Consider the points X, Y, Z with absolute barycentric coordinates:

$$X = (0, u, 1 - u),\ Y = (1 - v, 0, v),\ Z = (w, 1 - w, 0).$$

Let's denote the signed area of the $A_1A_2 \dots A_n$ polygon with the symbol $\sigma[A_1A_2 \dots A_n]$ and the area of the triangle ABC with Δ. Let $X'Y'Z'$ be another inscribed triangle in the triangle ABC:

$$X' = (0, u', 1 - u'),\ Y' = (1 - v', 0, v'),\ Z' = (w', 1 - w', 0).$$

2010 *Mathematics Subject Classification.* 97G40.

Key words and phrases. Barycentric coordinates, isotomic points.
Received: 24.06.2015. In revised form: 24.11.2015. Accepted: 10.02.2016.

2. The conditions under which two inscribed triangles have equal area

The area of the triangles XYZ and $X'Y'Z'$ are:

$$\sigma[XYZ] = \begin{vmatrix} 0 & u & 1-u \\ 1-v & 0 & v \\ w & 1-w & 0 \end{vmatrix} \Delta = [uvw + (1-u)(1-v)(1-w)]\Delta,$$

$$\sigma[X'Y'Z'] = \begin{vmatrix} 0 & u' & 1-u' \\ 1-v' & 0 & v' \\ w' & 1-w' & 0 \end{vmatrix} \Delta = [u'v'w' + (1-u')(1-v')(1-w')]\Delta.$$

The triangles XYZ and $X'Y'Z'$ have an equal area if

$$|uvw + (1-u)(1-v)(1-w)| = |u'v'w' + (1-u')(1-v')(1-w')|. \quad (2.1)$$

If we consider fix the triangle $X'Y'Z'$ and the pair of variables (v, w), then from condition (2.1) we obtain two values for the variable u. This signifies that there are an infinite number of inscribed triangles which have the same area as the fixed $X'Y'Z'$ triangle. Several examples follow:

Let A', B', C' be the midpoint of the side BC, CA, AB, and $X' = A' = \left(0, \frac{1}{2}, \frac{1}{2}\right)$, $Y' = B' = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$, $Z' = C' = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. Following $u' = v' = w' = \frac{1}{2}$ and $\sigma[X'Y'Z'] = \sigma[A'B'C'] = \frac{\Delta}{4}$.

2.1 If $v = \frac{2}{3}$ and $w = \frac{1}{4}$, then $u = 0$ or $u = 6$. In this case $X = (0, 0, 1) = C$ or $X' = (0, 6, -5)$ and $Y = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$, $Z = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$. It can easily verified that $\sigma[XYZ] = \frac{\Delta}{4} = \sigma[X'Y'Z']$ and $\sigma[X'YZ] = -\frac{\Delta}{4} = -\sigma[X'Y'Z']$.

2.2 If $v = 2$ and $w = 1$, then $u = \frac{1}{8}$ or $u = -\frac{1}{8}$. In this case $X_1 = \left(0, \frac{1}{8}, \frac{7}{8}\right)$ or $X_1^* = \left(0, -\frac{1}{8}, \frac{9}{8}\right)$, $Y_1 = (-1, 0, 2)$, $Z_1 = (1, 0, 0) = A$, $\sigma[X_1Y_1Z_1] = \frac{\Delta}{4} = \sigma[X'Y'Z']$ and $\sigma[X_1^*Y_1Z_1] = -\frac{\Delta}{4} = -\sigma[X'Y'Z']$.

2.3 If $v = -4$ and $w = 2$, then $u = -\frac{7}{4}$ or $u = -\frac{19}{12}$. In this case $X_2 = \left(0, -\frac{7}{4}, \frac{11}{4}\right)$ or $X_2^* = \left(0, -\frac{19}{12}, \frac{31}{12}\right)$, $Y_2 = (5, 0, -4)$, $Z_2 = (2, -1, 0)$, $\sigma[X_2Y_2Z_2] = \frac{\Delta}{4} = \sigma[X'Y'Z']$ and $\sigma[X_2^*Y_2Z_2] = -\frac{\Delta}{4} = -\sigma[X'Y'Z']$.

Now we represent the triangles XYZ, $X'Y'Z'$ and $X_1^*Y_1Z_1$ (Figure 1). If the point X is on the line BC and $t = \frac{BX}{XC}$, then the absolute barycentric coordinates of X can be obtained as a convex combination of B and C: $X = \left(1 - \frac{t}{t+1}\right)B + \frac{t}{t+1}C = \frac{1}{t+1}B + \frac{t}{t+1}C$ and $BX = \frac{t}{t+1}BC$,

$XC = \dfrac{1}{t+1}BC$. This result is used also in the case of XYZ and $X_1^*Y_1Z_1$ triangles:

$$Y = \frac{2}{3}C + \frac{1}{3}A, \quad \frac{CY}{YA} = \frac{1}{2}, \; CY = \frac{1}{3}CA, \quad YA = \frac{2}{3}CA,$$

$$Z = \frac{1}{4}A + \frac{3}{4}B, \quad \frac{AZ}{ZB} = 3, \quad AZ = \frac{3}{4}AB, \quad ZB = \frac{1}{4}AB,$$

$$X_1^* = -\frac{1}{8}B + \frac{9}{8}C, \quad \frac{BX_1^*}{X_1^*C} = -9, \quad BX_1^* = \frac{9}{8}BC, \quad X_1^*C = -\frac{1}{8}BC,$$

$$Y_1 = 2C - A, \quad \frac{CY_1}{Y_1A} = -\frac{1}{2}, \quad CY_1 = -CA, \quad Y_1A = 2CA.$$

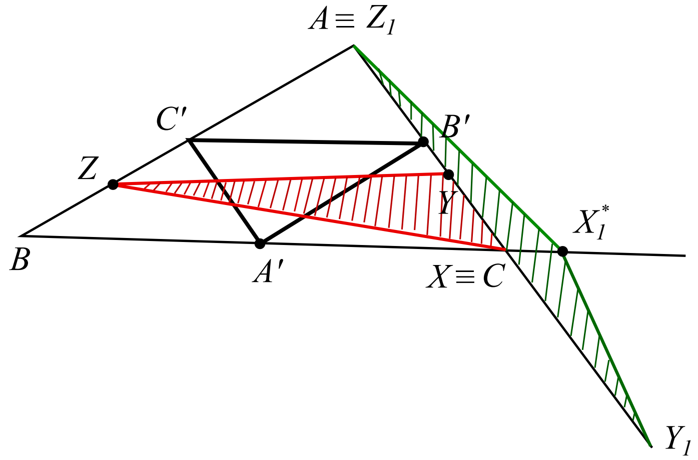

Figure 1

3. Construction of the inscribed equal area triangles

Theorem 3.1. *If X', Y', Z' is the symmetric of X, Y, Z with respect to A', B', C', then*

$$\sigma[XYZ] = (1 - u - v - w + vw + wu + uv)\Delta = \sigma[X'Y'Z'], \tag{3.1}$$

$$\sigma[X'YZ] = (u + vw - wu - uv)\Delta = \sigma[XY'Z'], \tag{3.2}$$

$$\sigma[XY'Z] = (v - vw + wu - uv)\Delta = \sigma[X'YZ'], \tag{3.3}$$

$$\sigma[XYZ'] = (w - vw - wu + uv)\Delta = \sigma[X'Y'Z]. \tag{3.4}$$

Proof. In this case $X' = (0, 1-u, u)$, $Y' = (v, 0, 1-v)$, $Z' = (1-w, w, 0)$ (Figure 2). Consequently,

$$\sigma[X'Y'Z'] = \begin{vmatrix} 0 & 1-u & u \\ v & 0 & 1-v \\ 1-w & w & 0 \end{vmatrix}\Delta = [uvw + (1-u)(1-v)(1-w)]\Delta = \sigma[XYZ],$$

$$\sigma[X'YZ] = \begin{vmatrix} 0 & 1-u & u \\ 1-v & 0 & v \\ w & 1-w & 0 \end{vmatrix}\Delta = \begin{vmatrix} 0 & u & 1-u \\ v & 0 & 1-v \\ 1-w & w & 0 \end{vmatrix}\Delta = \sigma[XY'Z'].$$

□

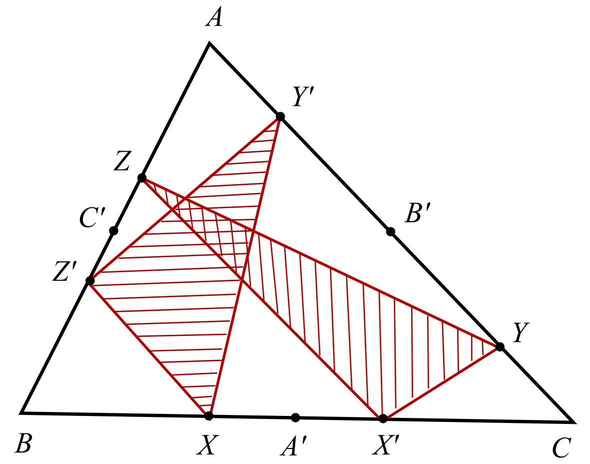

Figure 2

3.1 Consider the following numerical examples: $X = (0, 0, 1) = C$, $Y = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$, $Z = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$. Therefore $X' = (0, 1, 0) = B$, $Y = \left(\frac{2}{3}, 0, \frac{1}{3}\right)$, $Z' = \left(\frac{3}{4}, \frac{1}{4}, 0\right)$ (Figure 3).

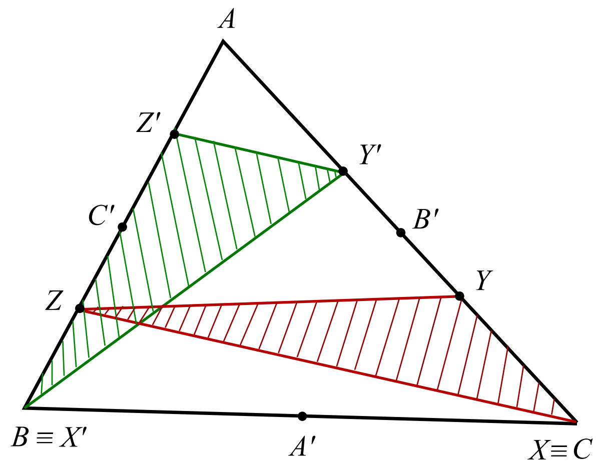

Figure 3

Now we calculate elementary the area of the triangles XYZ, $X'Y'Z'$, $X'YZ$, $XY'Z'$:

$$\begin{aligned}\sigma[XYZ] &= \Delta - \sigma[AZY] - \sigma[BCZ] = \Delta - \frac{1}{2}\frac{2b}{3}\frac{3c}{4}\sin A - \frac{1}{2}a\frac{c}{4}\sin B\\ &= \Delta - \frac{\Delta}{2} - \frac{\Delta}{4} = \frac{\Delta}{4},\\ \sigma[X'Y'Z'] &= \Delta - \sigma[AZ'Y'] - \sigma[BCY'] = \Delta - \frac{1}{2}\frac{b}{3}\frac{c}{4}\sin A - \frac{1}{2}a\frac{2b}{3}\sin C\\ &= \Delta - \frac{\Delta}{12} - \frac{2\Delta}{3} = \frac{\Delta}{4},\end{aligned}$$

$$\sigma[X'YZ] = \Delta - \sigma[AZY] - \sigma[BCY] = \Delta - \frac{1}{2}\frac{2b}{3}\frac{3c}{4}\sin A - \frac{1}{2}a\frac{b}{3}\sin C$$
$$= \Delta - \frac{\Delta}{2} - \frac{\Delta}{3} = \frac{\Delta}{6},$$
$$\sigma[XY'Z'] = \Delta - \sigma[AZ'Y'] - \sigma[BCZ'] = \Delta - \frac{1}{2}\frac{b}{3}\frac{c}{4}\sin A - \frac{1}{2}a\frac{3c}{4}\sin B$$
$$= \Delta - \frac{\Delta}{12} - \frac{3\Delta}{4} = \frac{\Delta}{6}.$$

Similarly we obtain: $\sigma[XY'Z] = \sigma[X'YZ'] = \dfrac{\Delta}{2}$ and $\sigma[XYZ'] = \sigma[X'Y'Z] = \dfrac{\Delta}{12}$.

3.2 Let a, b, c be the sidelength of triangle ABC and s its semiperimeter: $2s = a + b + c$. The touchpoints of the incircle with the side BC, CA, AB are $D = \left(0, \dfrac{s-c}{a}, \dfrac{s-b}{a}\right)$, $E = \left(\dfrac{s-c}{b}, 0, \dfrac{s-a}{b}\right)$, $F = \left(\dfrac{s-b}{c}, \dfrac{s-a}{c}\right)$. The A-excircle, B-excircle and C-excircle touch the side BC, CA, AB in the points $D' = \left(0, \dfrac{s-b}{a}, \dfrac{s-c}{a}\right)$, $E' = \left(\dfrac{s-a}{b}, 0, \dfrac{s-c}{b}\right)$, respectively $F' = \left(\dfrac{s-a}{c}, \dfrac{s-b}{c}, 0\right)$ (Figure 4).

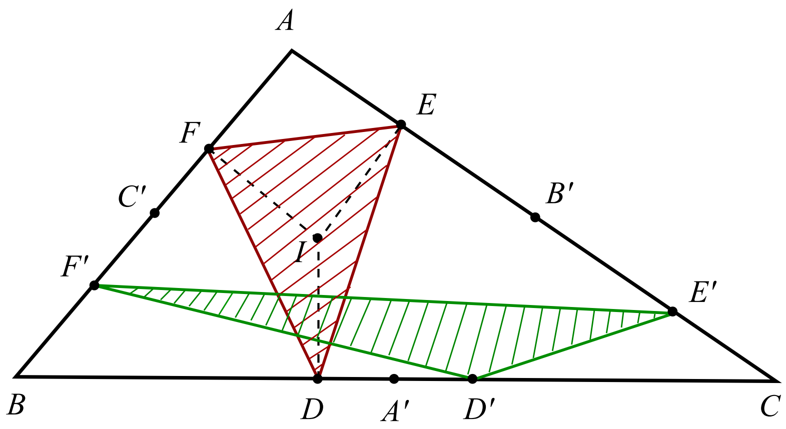

Figure 4

Corollary 3.1. *The pairs of points $(D, D'), (E, E'), (F, F')$ are symmetric with respect to the midpoints A', B', C' of the side BC, CA, AB and consequently*

$$\sigma[DEF] = \sigma[D'E'F'] = \frac{2(s-a)(s-b)(s-c)}{abc}\Delta, \tag{3.5}$$
$$\sigma[D'EF] = \sigma[DE'F'] = \frac{(b+c-a)(a^2+b^2+c^2-2bc)}{4abc}\Delta \tag{3.6}$$
$$\sigma[DE'F] = \sigma[D'EF'] = \frac{(c+a-b)(a^2+b^2+c^2-2ca)}{4abc}\Delta, \tag{3.7}$$
$$\sigma[DEF'] = \sigma[D'E'F] = \frac{(a+b-c)(a^2+b^2+c^2-2ab)}{4abc}\Delta. \tag{3.8}$$

We deduce the equality (3.5):

$$\sigma[DEF] = \sigma[D'E'F']$$
$$= \left(1 - \frac{s-c}{a} - \frac{s-a}{b} - \frac{s-b}{c} + \frac{(s-a)(s-b)}{bc} + \frac{(s-b)(s-c)}{ca} + \frac{(s-c)(s-a)}{ab}\right)\Delta$$
$$= \left(\frac{a+c-s}{a} - \frac{s-a}{b} - \frac{s-b}{c} + \frac{(s-a)(s-b)}{bc} + \frac{(s-b)(s-c)}{ca} + \frac{(s-c)(s-a)}{ab}\right)\Delta$$
$$= \left(\frac{s-b}{a} - \frac{s-a}{b} - \frac{s-b}{c} + \frac{(s-a)(s-b)}{bc} + \frac{(s-b)(s-c)}{ca} + \frac{(s-c)(s-a)}{ab}\right)\Delta$$
$$= \left(\frac{s-b}{a}\,\frac{s}{c} - \frac{s-a}{b} - \frac{s-b}{c} + \frac{(s-a)(s-b)}{bc} + \frac{s-c)(s-a)}{ab}\right)\Delta$$
$$= \left(\frac{(s-a)(s-b)}{ca} - \frac{s-a}{b} + \frac{(s-a)(s-b)}{bc} + \frac{(s-c)(s-a)}{ab}\right)\Delta$$
$$= (s-a)\left(\frac{s-b}{ca} - \frac{1}{b} + \frac{s-b}{bc} + \frac{s-c}{ab}\right)\Delta$$
$$= (s-a)\left(\frac{s-b}{ca} + \frac{s-b}{bc} + \frac{s-c-a}{ab}\right)\Delta$$
$$= (s-a)(s-b)\frac{a+b-c}{abc}\Delta = \frac{2(s-a)(s-b)(s-c)}{abc}\Delta.$$

3.3 The excircles touch the side BC, CA, AB in several points

$$P = \left(0, \frac{s}{a}, -\frac{s-a}{a}\right),\ Q = \left(-\frac{s-b}{b}, 0, \frac{s}{b}\right),\ R = \left(\frac{s}{c}, -\frac{s-c}{c}, 0\right),$$

$$P' = \left(0, -\frac{s-a}{a}, \frac{s}{a}\right),\ Q' = \left(\frac{s}{b}, 0, -\frac{s-b}{b}\right),\ R' = \left(-\frac{s-c}{c}, \frac{s}{c}, 0\right),$$

(Figure 5).

Corollary 3.2. *The pairs of points $(P,P'), (Q,Q'), (R,R')$ are symmetric with respect to the midpoints A', B', C' of the side BC, CA, AB and consequently*

$$\sigma[PQR] = \sigma[P'Q'R'] = \frac{2s^3 - (bc+ca+ab)s + abc}{abc}\Delta \tag{3.9}$$

$$\sigma[P'QR] = \sigma[PQ'R'] = \frac{(a+b+c)(a^2-b^2-c^2)}{4abc}\Delta \tag{3.10}$$

$$\sigma[PQ'R] = \sigma[P'QR'] = \frac{(a+b+c)(b^2-c^2-a^2)}{4abc}\Delta \tag{3.11}$$

$$\sigma[PQR'] = \sigma[P'Q'R] = \frac{(a+b+c)(c^2-a^2-b^2)}{4abc}\Delta. \tag{3.12}$$

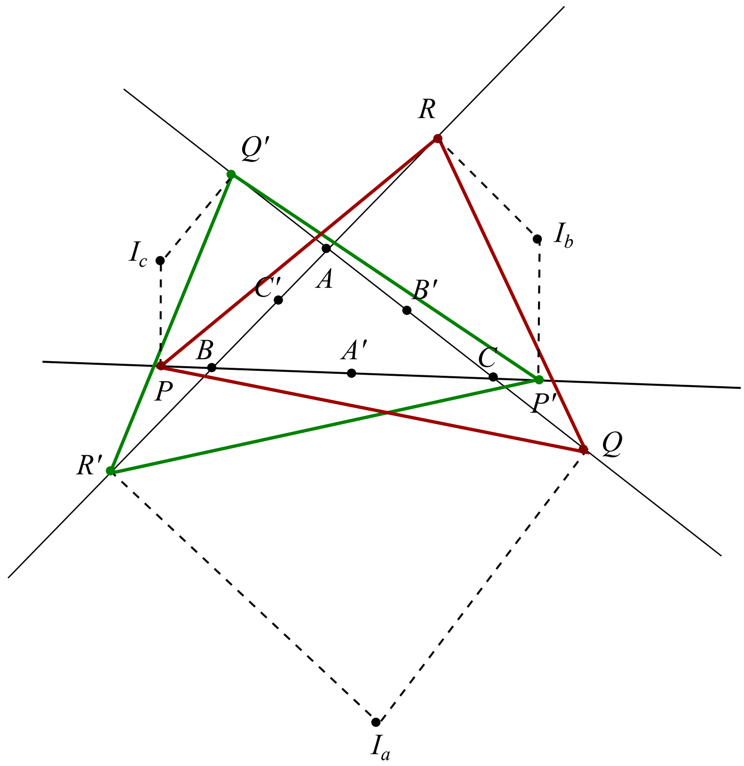

Figure 5

Here we deduce the equality (3.10):

$$\begin{aligned}\sigma[P'QR] = \sigma[PQ'R'] &= \left(\frac{s}{a} + \frac{s}{b}\frac{s}{c} - \frac{s}{c}\frac{s}{a} - \frac{s}{a}\frac{s}{b}\right)\Delta \\ &= \frac{s}{abc}[bc - s(b+c-a)]\Delta \\ &= \frac{s}{2abc}[2bc - (b+c+a)(b+c-a)]\Delta \\ &= \frac{(a+b+c)(a^2-b^2-c^2)}{4abc}\Delta\end{aligned}$$

(Figure 5).

4. Inscribed equal area triangles and the isotomic points

Let L, M, N be three points on the sides of the triangle ABC: $L \in BC$, $M \in CA$, $N \in AB$. It is well-known that if the triangles LMN and ABC are perspectives and if L', M', N' is the symmetric of L, M, N with respect to A', B', C', then the triangles $L'M'N'$ and ABC are perspectives too and conversely. Note their perspectors with P and P': $P = AL \cap BM \cap CN$ and $P' = AL' \cap BM' \cap CN'$. The P and P' points are called *isotomic points.*

Theorem 4.1. *If the barycentric coordinates of the point P are $(\alpha : \beta : \gamma)$, $\alpha\beta\gamma \neq 0$, then the barycentric coordinates of its isotomic point P' are*

$\left(\frac{1}{\alpha} : \frac{1}{\beta} : \frac{1}{\gamma}\right)$ *and*

$$\sigma[LMN] = \frac{2\alpha\beta\gamma}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}\,\Delta = \sigma[L'M'N'], \tag{4.1}$$

$$\sigma[L'MN] = \frac{\alpha(\beta^2+\gamma^2)}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}\,\Delta = \sigma[LM'N'], \tag{4.2}$$

$$\sigma[LM'N] = \frac{\beta(\gamma^2+\alpha^2)}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}\,\Delta = \sigma[L'MN'], \tag{4.3}$$

$$\sigma[LMN'] = \frac{\gamma(\alpha^2+\beta^2)}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}\,\Delta = \sigma[L'M'N]. \tag{4.4}$$

Proof. In this case the barycentric coordinates of the points L, M, N and L', M', N' are

$$L = (0;\beta:\gamma), \quad M = (\alpha:0:\gamma), \quad N = (\alpha:\beta:0),$$

$$L' = (0:\gamma:\beta), \quad M' = (\gamma:0:\alpha), \quad N' = (\beta:\alpha:0).$$

The absolute barycentric coordinates of this points are

$$L = \left(0, \frac{\beta}{\beta+\gamma}, \frac{\gamma}{\beta+\gamma}\right), \; M = \left(\frac{\alpha}{\gamma+\alpha}, 0, \frac{\gamma}{\gamma+\alpha}\right), \; N = \left(\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}, 0\right),$$

$$L' = \left(0, \frac{\gamma}{\beta+\gamma}, \frac{\beta}{\beta+\gamma}\right), \; M' = \left(\frac{\gamma}{\gamma+\alpha}, 0, \frac{\alpha}{\gamma+\alpha}\right), \; N' = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}, 0\right).$$

Consequently

$$\begin{aligned}\sigma[LMN] &= \frac{1}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}\begin{vmatrix}0 & \beta & \gamma\\ \alpha & 0 & \gamma\\ \alpha & \beta & 0\end{vmatrix}\Delta\\ &= \frac{2\alpha\beta\gamma}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}\Delta = \sigma[L'M'N'],\\ \sigma[L'MN] &= \frac{1}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}\begin{vmatrix}0 & \gamma & \beta\\ \alpha & 0 & \gamma\\ \alpha & \beta & 0\end{vmatrix}\Delta\\ &= \frac{\alpha(\beta^2+\gamma^2)}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}\Delta = \sigma[LM'N'].\end{aligned}$$

□

Henceforth we consider pairs of isotomic points formed with triangles centers [3]. Let's denote the pairs of these triangle centers with $X(i)$ and $X(j)$. Let's consider $L_i = BC \cap AX(i)$, $M_i = CA \cap BX(i)$, $N_i = AB \cap CX(i)$. We will write the formulas (4.1)–(4.4) for some particular cases.

4.1 The isotomic conjugate of the incenter $X(1) = (a : b : c)$ is the center $X(75)$:

$$\sigma[L_1M_1N_1] = \sigma[L_{75}M_{75}N_{75}] = \frac{2abc}{(b+c)(c+a)(a+b)}\,\Delta, \tag{4.5}$$

$$\sigma[L_{75}M_1N_1] = \sigma[L_1M_{75}N_{75}] = \frac{a(b^2+c^2)}{(b+c)(c+a)(a+b)}\,\Delta, \tag{4.6}$$

$$\sigma[L_1M_{75}N_1] = \sigma[L_{75}M_1N_{75}] = \frac{b(c^2+a^2)}{(b+c)(c+a)(a+b)}\,\Delta, \tag{4.7}$$

$$\sigma[L_1M_1N_{75}] = \sigma[L_{75}M_{75}N_1] = \frac{c(a^2+b^2)}{(b+c)(c+a)(a+b)}\,\Delta. \tag{4.8}$$

4.2 The isotomic conjugate of the centroid $X(2) = (1 : 1 : 1)$ is the centroid itself:

$$\sigma[L_2M_2N_2] = \sigma[A'B'C'] = \frac{\Delta}{4}\,. \tag{4.9}$$

4.3 The isotomic conjugate of the circumcenter $X(3) = (a\cos A : b\cos B : c\cos C)$ is the center $X(264)$:

$$\begin{aligned}\sigma[L_3M_3N_3] &= \sigma[L_{264}M_{264}N_{264}] \\ &= \frac{2abc\cos A\cos B\cos C}{(b\cos B+c\cos C)(c\cos C+a\cos A)(a\cos A+b\cos B)}\,\Delta,\end{aligned} \tag{4.10}$$

$$\begin{aligned}\sigma[L_{264}M_3N_3] &= \sigma[L_3M_{264}N_{264}] \\ &= \frac{a\cos A(b^2\cos^2 B+c^2\cos^2 C)}{(b\cos B+c\cos C)(c\cos C+a\cos A)(a\cos A+b\cos B)}\,\Delta,\end{aligned} \tag{4.11}$$

$$\begin{aligned}\sigma[L_3M_{264}N_3] &= \sigma[L_{264}M_3N_{264}] \\ &= \frac{b\cos B(c^2\cos^2 C+a^2\cos^2 A)}{(b\cos B+c\cos C)(c\cos C+a\cos A)(a\cos A+b\cos B)}\,\Delta,\end{aligned} \tag{4.12}$$

$$\begin{aligned}\sigma[L_3M_3N_{264}] &= \sigma[L_{264}M_{264}N_3] \\ &= \frac{c\cos C(a^2\cos^2 A+b^2\cos^2 B)}{(b\cos B+c\cos C)(c\cos C+a\cos A)(a\cos A+b\cos B)}\,\Delta.\end{aligned} \tag{4.13}$$

4.4 The isotomic conjugate of the orthocenter
$X(4) = (a\cos B\cos C : b\cos C\cos A : c\cos A\cos B)$ is the center $X(69)$:

$$\sigma[L_4M_4N_4] = \sigma[L_{69}M_{69}N_{69}] = 2\cos A\cos B\cos C\cdot\Delta, \tag{4.14}$$

(the area of the orthic triangle)

$$\sigma[L_{69}M_4N_4] = \sigma[L_4M_{69}N_{69}] = \frac{\cos A(b^2\cos^2 C+c^2\cos^2 B)}{bc}\,\Delta, \tag{4.15}$$

$$\sigma[L_4M_{69}N_4] = \sigma[L_{69}M_4N_{69}] = \frac{\cos B(c^2\cos^2 A+a^2\cos^2 C)}{ca}\,\Delta, \tag{4.16}$$

$$\sigma[L_4M_4N_{69}] = \sigma[L_{69}M_{69}N_4] = \frac{\cos C(a^2\cos^2 B+b^2\cos^2 A)}{ab}\,\Delta. \tag{4.17}$$

4.5 The isotomic conjugate of the symmedian point $X(6) = (a^2 : b^2 : c^2)$ is the 3^{rd} Brocard point $X(76)$:

$$\sigma[L_6M_6N_6] = \sigma[L_{76}M_{76}N_{76}] = \frac{2a^2b^2c^2}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}\,\Delta, \tag{4.18}$$

$$\sigma[L_{76}M_6N_6] = \sigma[L_6M_{76}N_{76}] = \frac{a^2(b^4+c^4)}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}\,\Delta, \tag{4.19}$$

$$\sigma[L_6M_{76}N_6] = \sigma[L_{76}M_6N_{76}] = \frac{b^2(c^4+a^4)}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}\,\Delta \tag{4.20}$$

$$\sigma[L_6M_6N_{76}] = \sigma[L_{76}M_{76}N_6] = \frac{c^2(a^4+b^4)}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}\,\Delta. \tag{4.21}$$

4.6 The isotomic conjugate of the Gergonne point $X(7)$ is the Nagel point $X(8) = (b+c-a : c+a-b : a+b-c)$ (see Corollary 3.1):

$$\sigma[L_7M_7N_7] = \sigma[L_8M_8N_8] = \sigma[DEF] = \sigma[D'E'F'] = \frac{2(s-a)(s-b)(s-c)}{abc}\,\Delta, \tag{4.22}$$

$$\sigma[L_8M_7N_7] = \sigma[L_7M_8N_8] = \sigma[D'EF] = \sigma[DE'F'] = \frac{(b+c-a)(a^2+b^2+c^2-2bc)}{4abc}\,\Delta, \tag{4.23}$$

$$\sigma[L_7M_8N_7] = \sigma[L_8M_7N_8] = \sigma[DE'F] = \sigma[D'EF'] = \frac{(c+a-b)(a^2+b^2+c^2-2ca)}{4abc}\,\Delta, \tag{4.24}$$

$$\sigma[L_7M_7N_8] = \sigma[L_8M_8N_7] = \sigma[DEF'] = \sigma[D'E'F] = \frac{(a+b-c)(a^2+b^2+c^2-2ab)}{4abc}\,\Delta. \tag{4.25}$$

4.7 The isotomic conjugate of the Spieker center $X(10) = (b+c : c+a : a+b)$ is the center $X(86)$:

$$\sigma[L_{10}M_{10}N_{10} = \sigma[L_{86}M_{86}N_{86}] = \frac{2(b+c)(c+a)(a+b)}{(b+c+2a)(c+a+2b)(a+b+2c)}\,\Delta, \tag{4.26}$$

$$\sigma[L_{86}M_{10}N_{10}] = \sigma[L_{10}M_{86}N_{86}] = \frac{(b+c)(b^2+c^2+2a^2+2ab+2ca)}{(b+c+2a)(c+a+2b)(a+b+2c)}\,\Delta, \tag{4.27}$$

$$\sigma[L_{10}M_{86}N_{10}] = \sigma[L_{86}M_{10}N_{86}] = \frac{(c+a)(c^2+a^2+2b^2+2bc+2ab)}{(b+c+2a)(c+a+2b)(a+b+2c)}\,\Delta, \tag{4.28}$$

$$\sigma[L_{10}M_{10}N_{86}] = \sigma[L_{86}M_{86}N_{10}] = \frac{(a+b)(a^2+b^2+2c^2+2ca+2bc)}{(b+c+2a)(c+a+2b)(a+b+2c)}\,\Delta. \tag{4.29}$$

It is possible to continue this calculus with another pairs of isotomic points.

5. Properties of inscribed triangles in a triangle

Theorem 5.1. *If X', Y', Z' are the symmetrics of X, Y, Z with respect to A', B', C' then the sum of the areas of the triangles XYZ, $X'YZ$, $XY'Z$, XYZ' is equal with the area of the triangle ABC, i.e.*

$$\sigma[XYZ] + \sigma[X'YZ] + \sigma[XY'Z] + \sigma[XYZ'] = \sigma[ABC]. \tag{5.1}$$

$$\sigma[X'Y'Z'] + \sigma[XY'Z'] + \sigma[X'YZ'] + \sigma[X'Y'Z] = \sigma[ABC]. \tag{5.2}$$

Proof. From the formula (3.1) we obtain:

$$\sigma[XYZ] + (u + v + w - vw - wu - uv)\Delta = \sigma[ABC].$$

Summing equality (3.2), (3.3) and (3.4) we obtain:

$$\sigma[X'YZ] + \sigma[XY'Z] + \sigma[XYZ'] = (u + v + w - vw - wu - uv)\Delta.$$

□

We verify the equality (5.1) for the triangles from 3.1:

$$\sigma[XYZ]+\sigma[X'YZ]+\sigma[XY'Z]+\sigma[XYZ'] = \frac{\Delta}{4} + \frac{\Delta}{6} + \frac{\Delta}{2} + \frac{\Delta}{12} = \sigma[ABC].$$

Let's verify another example: consider the inscribed triangle $X_2Y_2Z_2$ from 2.3, where

$X_2 = \left(0, -\frac{7}{4}, \frac{11}{4}\right)$, $Y_2 = (5, 0, -4)$, $Z_2 = (2, -1, 0)$ and

$X'_2 = \left(0, \frac{11}{4}, -\frac{7}{4}\right)$, $Y'_2 = (-4, 0, 5)$, $Z'_2 = (-1, 2, 0)$.

The areas of the triangles $X'_2Y_2Z_2$, $X_2Y'_2Z_2$ and $X_2Y_2Z'_2$ are

$$\sigma[X'_2Y_2Z_2] = \begin{vmatrix} 0 & \frac{11}{4} & -\frac{7}{4} \\ 5 & 0 & -4 \\ 2 & -1 & 0 \end{vmatrix} \Delta = \left(-22 + \frac{35}{4}\right)\Delta = -\frac{53}{4}\Delta,$$

$$\sigma[X_2Y'_2Z_2] = \begin{vmatrix} 0 & -\frac{7}{4} & \frac{11}{4} \\ -4 & 0 & 5 \\ 2 & -1 & 0 \end{vmatrix} \Delta = \left(11 - \frac{35}{2}\right)\Delta = -\frac{13}{2}\Delta,$$

$$\sigma[X_2Y_2Z'_2] = \begin{vmatrix} 0 & -\frac{7}{4} & \frac{11}{4} \\ 5 & 0 & -4 \\ -1 & 2 & 0 \end{vmatrix} \Delta = \left(-7 + \frac{110}{4}\right)\Delta = \frac{41}{2}\Delta.$$

So

$$\sigma[X_2Y_2Z_2] + \sigma[X'_2Y_2Z_2] + \sigma[X_2Y'_2Z_2] + \sigma[X_2Y_2Z'_2]$$
$$= \frac{\Delta}{4} - \frac{53\Delta}{4} - \frac{13\Delta}{2} + \frac{41\Delta}{2} = \sigma[ABC].$$

Theorem 5.2. *If* X', Y', Z' *is the symmetric of* X, Y, Z *with respect to* A', B', C', *then*

$$\sigma[X'YZ] = \sigma[XY'Z'] = \sigma[CYX] + (vw - wu)\Delta, \tag{5.3}$$

$$\sigma[XY'Z] = \sigma[X'YZ'] = \sigma[AZY] + (wu - uv)\Delta, \tag{5.4}$$

$$\sigma[XYZ'] = \sigma[X'Y'Z] = \sigma[BXZ] + (uv - vw)\Delta. \tag{5.5}$$

Proof. The areas of the triangles AZY, BXZ and CYX are

$$\sigma[AZY] = \begin{vmatrix} 1 & 0 & 0 \\ w & 1-w & 0 \\ 1-v & 0 & v \end{vmatrix} \Delta = v(1-w)\Delta = (v - vw)\Delta,$$

$$\sigma[BXZ] = \begin{vmatrix} 0 & 1 & 0 \\ 0 & u & 1-u \\ w & 1-w & 0 \end{vmatrix} \Delta = w(1-u)\Delta = (w - wu)\Delta,$$

$$\sigma[CYX] = \begin{vmatrix} 0 & 0 & 1 \\ 1-v & 0 & v \\ 0 & u & 1-u \end{vmatrix} \Delta = u(1-v)\Delta = (u - uv)\Delta.$$

□

Corollary 5.1. *If* X', Y', Z' *is the symmetric of* X, Y, Z *with respect to* A', B', C' *then*

$$\sigma[X'YZ] + \sigma[XY'Z] + \sigma[XYZ'] = \sigma[AZY] + \sigma[BXZ] + \sigma[CYX], \tag{5.6}$$

$$\sigma[XY'Z'] + \sigma[X'YZ'] + \sigma[X'Y'Z] = \sigma[AZY] + \sigma[BXZ] + \sigma[CYX]. \tag{5.7}$$

References

[1] Gómez, B.H., *The notable configuration of inscribed equilateral triangles in a triangle*, Elemente de Mathematik, **57** (2002), No. 1, 26–31.

[2] *Equilateral triangle inscribed in a triangle*, http://math.stackexchange.com/questions/ /186432/equilateral-triangle-inscribed-in-a-triangle.

[3] Kimberling, C., *Encyclopedia of Triangle Centers – ETC*, http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

"CONSTANTIN BRÂNCUŞI" TECHNOLOGY LYCEUM
SATU MARE, ROMANIA
e-mail: d.sandor.kiss@gmail.com