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METRIC RELATIONS IN EXTANGENTIAL QUADRILATERALS

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Abstract. Formulas for several quantities in convex quadrilaterals that have an excircle are derived, including the exradius, the area, the lengths of the tangency chords and the angle between them. We also prove six necessary and sufficient conditions for an extangential quadrilateral to be a kite, and deduce five formulas for the area of cyclic extangential quadrilaterals.

1. Introduction

An *extangential quadrilateral* is a convex quadrilateral with an *excircle*, which means an external circle tangent to the extensions of all four sides (see Figure 1). It is well known that triangles always have three excircles, and their properties have been extensively studied for centuries. A convex quadrilateral can however at most have one excircle. The extangential quadrilateral appears very rarely in geometry textbooks and in advanced problem solving compared to the tangential quadrilateral (a quadrilateral with an incircle) and the more famous cyclic quadrilateral (a quadrilateral with a circumcircle).

In [9] and [11] we proved a total of fifteen characterizations of extangential quadrilaterals. We remind the reader that a convex quadrilateral $ABCD$ with consecutive sides $a = AB$, $b = BC$, $c = CD$ and $d = DA$ has an excircle outside the biggest of the vertices A and C if and only if (see [9, p.64])

$$a + b = c + d, \tag{1}$$

and an excircle outside the biggest of the vertices B and D if and only if

$$a + d = b + c. \tag{2}$$

Here we continue to study extangential quadrilaterals, and we will for instance prove some corresponding theorems to the ones concerning tangential quadrilaterals in [4], [5], [7] and [10].

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In the derivations of formulas in this paper there are always four possible cases since the excircle can be outside any of the four vertices (depending on if the right conditions on sides and angles hold). We will however in most of the proofs only consider one case, and will then assume that the excircle is outside the vertex C. The other cases can be dealt with in the same way, or it is simply a matter of relabelling the vertices and sides. The formulas in this paper are the same no matter which vertex the excircle is outside.

2. The exradius

In [9] we proved that the exradius ρ (the radius in the excircle) in an extangential quadrilateral with consecutive sides a, b, c, d is given by

$$\rho = \frac{K}{|a-c|} = \frac{K}{|b-d|} \tag{3}$$

where K is the area of the quadrilateral. In this section we shall derive two other formulas for the exradius, and in the next section we prove several formulas for the area of an extangential quadrilateral. We will also compare these formulas to the similar ones that hold in a tangential quadrilateral.

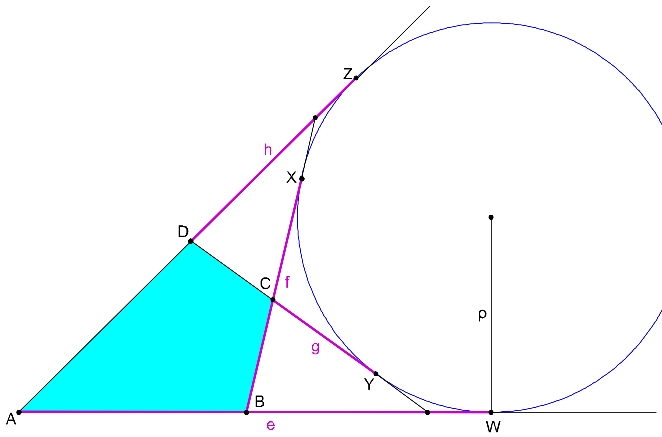

Figure 1. The tangent lengths $e = AW$, $f = BX$, $g = CY$, $h = DZ$

Let the excircle in an extangential quadrilateral $ABCD$ be tangent to the extensions of the sides AB, BC, CD, DA at W, X, Y, Z respectively, see Figure 1. We will call the distances $e = AW$, $f = BX$, $g = CY$, $h = DZ$ the *tangent lengths* of the extangential quadrilateral. Note that $AW = AZ$, $BX = BW$, $CY = CX$ and $DZ = DY$ according to the two tangent theorem.

Theorem 2.1. *In an extangential quadrilateral with consecutive sides a, b, c, d and tangent lengths e, f, g, h, the following statements are equivalent:*
(i) *The quadrilateral is a parallelogram.*
(ii) $a = c$
(iii) $b = d$
(iv) $e + g = f + h$

Proof. A quadrilateral is a parallelogram if and only if $a = c$ and $b = d$. Since $a = e - f$, $b = f - g$, $c = h - g$ and $d = e - h$ (if the excircle is outside of C, see Figure 1), we have

$$a - c = e - f + g - h = d - b.$$

Hence in an extangential quadrilateral

$$a = c \quad \Leftrightarrow \quad e + g = f + h \quad \Leftrightarrow \quad b = d,$$

so it is a parallelogram if and only if one pair of opposite sides are congruent. □

The inradius r in a tangential quadrilateral can be expressed in terms of the tangent lengths, i.e. the four distances e, f, g, h from the vertices to the points where the incircle is tangent to the sides. The formula is ([5, p.119])

$$r = \sqrt{\frac{efg + fgh + ghe + hef}{e + f + g + h}}. \tag{4}$$

Now we derive the formula for the exradius in terms of the tangent lengths. Note that two internal and two external angle bisectors are concurrent at the excenter J (the center in the excircle). These are u, x and v, y respectively in Figure 2.

Theorem 2.2. *An extangential quadrilateral with tangent lengths e, f, g, h has the exradius*

$$\rho = \sqrt{\frac{-efg + fgh - ghe + hef}{e - f + g - h}}.$$

The quadrilateral is a parallelogram in the case when the denominator is zero.

Proof. According to the sum of angles in a quadrilateral, $\frac{A}{2} + \frac{C}{2} = \pi - \left(\frac{B}{2} + \frac{D}{2}\right)$, so

$$\tan\left(\frac{A}{2} + \frac{C}{2}\right) = -\tan\left(\frac{B}{2} + \frac{D}{2}\right).$$

Using the addition formula, we get

$$\frac{\tan\frac{A}{2} + \tan\frac{C}{2}}{1 - \tan\frac{A}{2}\tan\frac{C}{2}} = -\frac{\tan\frac{B}{2} + \tan\frac{D}{2}}{1 - \tan\frac{B}{2}\tan\frac{D}{2}}$$

where $\tan\frac{A}{2} = \frac{\rho}{AW} = \frac{\rho}{e}$ and $\tan\frac{C}{2} = \frac{\rho}{CY} = \frac{\rho}{g}$ (see Figure 2). Thus

$$\frac{\frac{\rho}{e} + \frac{\rho}{g}}{1 - \frac{\rho}{e}\cdot\frac{\rho}{g}} = -\frac{\frac{f}{\rho} + \frac{h}{\rho}}{1 - \frac{f}{\rho}\cdot\frac{h}{\rho}}$$

since $\cot\frac{B}{2} = \tan\frac{\pi - B}{2} = \frac{\rho}{BX}$, so $\tan\frac{B}{2} = \frac{f}{\rho}$, and similar $\tan\frac{D}{2} = \frac{h}{\rho}$. This is equivalent to

$$\rho\left(\frac{1}{e} + \frac{1}{g}\right)\left(1 - \frac{fh}{\rho^2}\right) = -\frac{1}{\rho}(f + h)\left(1 - \frac{\rho^2}{eg}\right)$$

which by multiplying on both sides with ρ and expanding them yields

$$\rho^2\left(\frac{e + g}{eg}\right) - \frac{(e + g)fh}{eg} = -(f + h) + \frac{f + h}{eg}\rho^2.$$

Clearing the denominators, we get

$$\rho^2(e+g)-(e+g)fh=-eg(f+h)+(f+h)\rho^2.$$

This is equivalent to

$$\rho^2(e-f+g-h)=(e+g)fh-eg(f+h)$$

and the formula follows. It is not valid if $e-f+g-h=0$, in which case the quadrilateral is a parallelogram according to Theorem 2.1. Note that parallelograms can be considered to be extangential quadrilateral with infinite exradius according to (3), since they satisfy (1) and (2). □

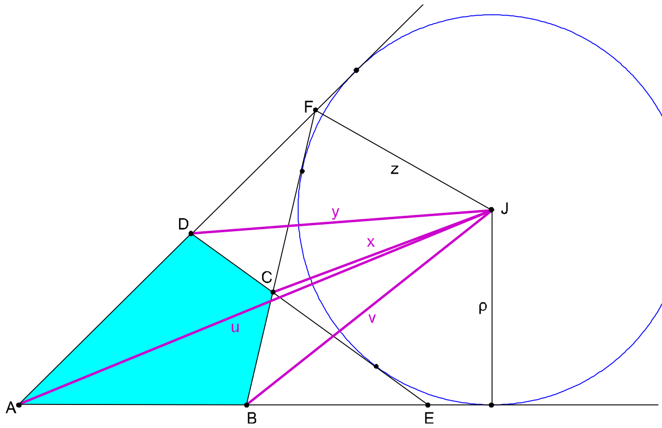

Figure 2. Distances u, v, x, y from the excenter to the vertices

It is noteworthy that the only difference between the formula for the inradius in a tangential quadrilateral and the exradius to an extangential quadrilateral is the signs on two of the tangent lengths. From this point of view we get the formula in Theorem 2.2 by making the changes $f\to -f$ and $h\to -h$ in (4). This symmetry between these two types of quadrilaterals will appear again.

In [4] we proved that the inradius in a tangential quadrilateral can be expressed in terms of the four distances u, v, x, y between the incenter and the vertices according to the beautiful formula

$$r=2\sqrt{\frac{(M-uvx)(M-vxy)(M-xyu)(M-yuv)}{uvxy(uv+xy)(ux+vy)(uy+vx)}} \tag{5}$$

where $M=\frac{1}{2}(uvx+vxy+xyu+yuv)$. Now we shall derive the corresponding formula for an extangential quadrilateral.

Theorem 2.3. *If u, v, x, y are the distances between the excenter and the vertices of an extangential quadrilateral, then its exradius is given by*

$$\rho=2\sqrt{\frac{(\eta-uvx)(\eta-vxy)(\eta-xyu)(\eta+yuv)}{uvxy(uv-xy)(ux-vy)(uy-vx)}}$$

where $\eta=\frac{1}{2}(uvx+vxy+xyu-yuv)$.

Proof. In an extangential quadrilateral $ABCD$ with excenter J, let the extensions of opposite sides BC and AD intersect at F. The distances from the excenter to the vertices are denoted $u=AJ$, $v=BJ$, $x=CJ$

and $y = DJ$. Also, let $z = FJ$. The excircle to the quadrilateral is also an excircle to the triangles ABF and CFD outside the sides BF and CF respectively (see Figure 2). Applying Lemma 3 in [4] to these two triangles yields that the exradius satisfies the two cubic equations

$$2xyz\rho^3 - (x^2y^2 + y^2z^2 + z^2x^2)\rho^2 + x^2y^2z^2 = 0, \tag{6}$$

$$2uvz\rho^3 - (u^2v^2 + v^2z^2 + z^2u^2)\rho^2 + u^2v^2z^2 = 0. \tag{7}$$

Comparing these two equations to (4) and (5) in [4], we see that (7) in this paper and (5) in [4] are identical, whereas (6) in this paper and (4) in [4] only differ regarding the sign on x (or y). Thus instead of going the long way and eliminating the common (uninteresting) distance z from the two cubic equations and solving the remaining equation to express ρ in terms of u, v, x, y (which can be done in the same way as in [4, §3]), we simply have to make the change $x \to -x$ in (5) to get the formula for ρ. Then we have

$$M = \tfrac{1}{2}(-uvx - vxy - xyu + yuv) = -\eta.$$

Hence the exradius is given by

$$\begin{aligned}\rho &= 2\sqrt{\frac{(M + uvx)(M + vxy)(M + xyu)(M - yuv)}{-uvxy(uv - xy)(-ux + vy)(uy - vx)}}\\ &= 2\sqrt{\frac{(-1)^4(\eta - uvx)(\eta - vxy)(\eta - xyu)(\eta + yuv)}{uvxy(uv - xy)(ux - vy)(uy - vx)}}\end{aligned}$$

and the formula follows. □

3. The area of an extangential quadrilateral

In this section we shall derive six formulas for the area of an extangential quadrilateral. The first expresses the area in terms of the tangent lengths. The similar formula for a tangential quadrilateral is

$$K = \sqrt{(e + f + g + h)(efg + fgh + ghe + hef)}, \tag{8}$$

which is a simple corollary to (4) since $K = (e + f + g + h)r$ in a tangential quadrilateral.

After the proof of Theorem 2.2 we noted that the formula for the exradius only differs from the formula for the inradius in a tangential quadrilateral by the signs of two of the tangent lengths. According to the next theorem, the same is true for the area formula for an extangential quadrilateral.

Theorem 3.1. *An extangential quadrilateral with tangent lengths e, f, g, h has the area*

$$K = \sqrt{(e - f + g - h)(-efg + fgh - ghe + hef)}$$

if it is not a parallelogram.[1]

[1]And thus neither of the special cases rhombus, rectangle nor a square.

Proof. According to (3), the area is $K = |a-c|\rho$ where a and c are two opposite sides of the extangential quadrilateral. Since $a = e-f$ and $c = h-g$ (if the excircle is outside C), Theorem 2.2 yields that

$$\begin{aligned} K &= |e-f+g-h|\sqrt{\frac{-efg+fgh-ghe+hef}{e-f+g-h}} \\ &= \sqrt{(e-f+g-h)(-efg+fgh-ghe+hef)}. \end{aligned}$$

If instead the excircle is outside of B, then we get the same expressions for a and c as above. In the cases it is outside of A or D, then $a = f-e$ and $c = g-h$. However in all cases the value of $|a-c|$ is the same. Hence so is the expression for the area of the extangential quadrilateral.

The formula is not valid in a parallelogram since then $e-f+g-h = 0$, which would give an area equal to zero. In fact, the tangent lengths are not well defined as finite numbers in a parallelogram, which explains why the area cannot be expressed in terms of them. □

In the next theorem we prove four area formulas for extangential quadrilateral that are identical to the similar ones for a tangential quadrilateral.[2] We will use each of these formulas at least once in the proofs further ahead.

Theorem 3.2. *The area of an extangential quadrilateral $ABCD$ with consecutive sides a, b, c, d and tangent lengths e, f, g, h is given by the formulas*
(i) $K = \sqrt{abcd}\sin\frac{A+C}{2}$
(ii) $K = \frac{1}{2}\sqrt{p^2q^2-(ac-bd)^2}$
(iii) $K = \frac{1}{2}|ac-bd|\tan\theta$
(iv) $K = \sqrt{abcd-(eg-fh)^2}$,
where p and q are the diagonals and θ is the acute angle between them.

Proof. (i) The area of a convex quadrilateral is given by Bretschneider's formula

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+C}{2}\right)}$$

where s is the semiperimeter (for a proof, see [2, p.27]). In an extangential quadrilateral we either have $a+b = c+d = s$ or $a+d = b+c = s$. In the first case $s-a = b$, $s-b = a$, $s-c = d$, $s-d = c$; and in the second case $s-a = d$, $s-b = c$, $s-c = b$, $s-d = a$. Either way, inserting these simplified expressions into Bretschneider's formula directly yields

$$K = \sqrt{abcd - abcd\cos^2\left(\frac{A+C}{2}\right)} = \sqrt{abcd}\sin\frac{A+C}{2}$$

where we used the Pythagorean trigonometric identity to get the last equality.

[2]By similar ones we mean in terms of the same quantities. In the case of a tangential quadrilateral, a proof of formula (i) can be found in [2, p.28], formula (ii) was proved in [7, p.165], to prove formula (iii) was a problem in [2, p.29], and formula (iv) was proved in [5, pp.127–128].

(ii) The two possible side characterizations $a+b=c+d$ and $a+d=b+c$ for extangential quadrilateral gives $a-c=d-b$ and $a-c=b-d$. In both cases, squaring them and rearranging the terms yields

$$a^2-b^2+c^2-d^2=2(ac-bd). \tag{9}$$

Using this equality in the formula of Staudt (see [12, p.35])

$$K=\tfrac{1}{4}\sqrt{4p^2q^2-(a^2-b^2+c^2-d^2)^2},$$

which gives the area of a convex quadrilateral, we get that an extangential quadrilateral has the area

$$K=\tfrac{1}{4}\sqrt{4p^2q^2-(2(ac-bd))^2}$$

and the second formula follows.

(iii) Here we use yet another formula for the area of a convex quadrilateral,

$$K=\tfrac{1}{4}\left|a^2-b^2+c^2-d^2\right|\tan\theta$$

(see [2, p.27]). Inserting (9) directly proves the third formula.

(iv) We rewrite Theorem 3.1 according to

$$\begin{aligned}K^2&=(e-f+g-h)(-efg+fgh-ghe+hef)\\&=(e-f)(g-f)(g-h)(e-h)-(eg-fh)^2,\end{aligned}$$

and leave it for the reader to verify this equality (the rewrite can be done in the same way as we did for a tangential quadrilateral in [5, p.127]). Hence we get

$$K^2=a(-b)(-c)d-(eg-fh)^2=abcd-(eg-fh)^2$$

since $b=f-g$ and $c=h-g$ when $a=e-f$ and $d=e-h$. □

Problem 1 on Quiz 2 at the China Team Selection Test in 2003 [1] was to prove that in a tangential quadrilateral $ABCD$ with sides a, b, c, d and incenter I, it holds that

$$AI\cdot CI+BI\cdot DI=\sqrt{abcd}. \tag{10}$$

Combining this with formula (i) in Theorem 3.2 (that also holds in a tangential quadrilateral), we have that the area of a tangential quadrilateral is given by the not so well-known formula

$$K=(AI\cdot CI+BI\cdot DI)\sin\frac{A+C}{2}.$$

Now we derive the corresponding formula for an extangential quadrilateral.

Theorem 3.3. *An extangential quadrilateral $ABCD$ with excenter J has the area*

$$K=|AJ\cdot CJ-BJ\cdot DJ|\sin\frac{A+C}{2}$$

if it is not a parallelogram.

Proof. Let e, f, g, h be the tangent lengths and ρ the exradius. If the excircle is outside of the vertex C, we have that $\sin\frac{A}{2}=\frac{\rho}{AJ}$, $\cos\frac{A}{2}=\frac{e}{AJ}$, $\sin\frac{C}{2}=\frac{\rho}{CJ}$, and $\cos\frac{A}{2}=\frac{g}{CJ}$ (see Figure 2). Also

$$\cos\frac{B}{2}=\sin\frac{\pi-B}{2}=\frac{\rho}{BJ},\qquad \sin\frac{B}{2}=\cos\frac{\pi-B}{2}=\frac{f}{BJ}$$

and in the same way $\cos\frac{D}{2} = \frac{\rho}{DJ}$ and $\sin\frac{D}{2} = \frac{h}{DJ}$. Then we have

$$\sin\frac{A+C}{2} = \sin\frac{A}{2}\cos\frac{C}{2} + \cos\frac{A}{2}\sin\frac{C}{2}$$

$$= \frac{\rho}{AJ}\cdot\frac{g}{CJ} + \frac{e}{AJ}\cdot\frac{\rho}{CJ} = \frac{\rho}{AJ\cdot CJ}(e+g) \tag{11}$$

and similar

$$\sin\frac{B+D}{2} = \frac{\rho}{BJ\cdot DJ}(f+h).$$

If the consecutive sides of the extangential quadrilateral are a, b, c, d, then

$$|a-c| = |e-f-(h-g)| = |e+g-(f+h)|. \tag{12}$$

Expressing the area of the quadrilateral in two different ways using (3) and Theorem 3.2 (i), we get

$$\rho|a-c| = \sqrt{abcd}\sin\frac{A+C}{2}.$$

Inserting (12) and (11) and dividing both sides by ρ yields

$$|e+g-(f+h)| = \frac{\sqrt{abcd}}{AJ\cdot CJ}(e+g). \tag{13}$$

By symmetry (or deriving it in the same way), we also have

$$|e+g-(f+h)| = \frac{\sqrt{abcd}}{BJ\cdot DJ}(f+h). \tag{14}$$

The last two equalities yields

$$AJ\cdot CJ - BJ\cdot DJ = \frac{\sqrt{abcd}}{|e+g-f-h|}(e+g-f-h).$$

Taking the absolute value of both sides, we have proved that in an extangential quadrilateral

$$\sqrt{abcd} = |AJ\cdot CJ - BJ\cdot DJ|.$$

Inserting this in Theorem 3.2 (i) results in our sixth area formula for an extangential quadrilateral. Regarding the exception for parallelograms, see the proof of the corollary below. □

We note that the same method can be used in a tangential quadrilateral to derive the equality (10).

Corollary 3.1. *In an extangential quadrilateral $ABCD$ with excenter J and tangent lengths e, f, g, h, it holds that*

$$\frac{e+g}{f+h} = \frac{AJ\cdot CJ}{BJ\cdot DJ}$$

if the quadrilateral is not a parallelogram.

Proof. This equality follows directly from (13) and (14).

In Theorem 2.1 we established that an extangential quadrilateral is a parallelogram if and only if $e+g = f+h$, which is equivalent to $AJ\cdot CJ = BJ\cdot DJ$. This is the reason why the formulas in Theorem 3.3 and this corollary are not valid for parallelograms, since then J is a point at infinity. We would get expressions like infinity minus infinity and infinity divided by infinity, which are not defined. □

4. The tangency chords

If the excircle to an extangential quadrilateral $ABCD$ is tangent to the extensions of the sides AB, BC, CD, DA at W, X, Y, Z respectively, then we will call the distances WY and XZ the *tangency chords* (see Figure 4) in comparison to the corresponding line segments in a tangential quadrilateral, which we studied in [5, pp.120–121].

Theorem 4.1. *The lengths of the tangency chords WY and XZ in an extangential quadrilateral with tangent lengths e, f, g, h are respectively*

$$k = \frac{2(-efg + fgh - ghe + hef)}{\sqrt{(e-f)(f+h)(h-g)(g+e)}},$$

$$l = \frac{2(-efg + fgh - ghe + hef)}{\sqrt{(e-h)(h+f)(f-g)(g+e)}}.$$

Proof. Twice before we have seen that the only difference between formulas in terms of the tangent lengths in tangential and extangential quadrilaterals are the signs of f and h. We use that symmetry now to quickly get the formulas in this theorem. The lengths of the tangency chords in a tangential quadrilateral are according to [5, p.120]

$$k = \frac{2(efg + fgh + ghe + hef)}{\sqrt{(e+f)(f+h)(h+g)(g+e)}},$$

$$l = \frac{2(efg + fgh + ghe + hef)}{\sqrt{(e+h)(h+f)(f+g)(g+e)}}.$$

Making the changes $f \to -f$ and $h \to -h$ in these formulas directly proves this theorem after a little rewrite of the radicand. □

Another (longer) way to derive the two formulas for an extangential quadrilateral is to mimic the method used in the tangential case in [5].

For the quotient of the tangency chords we have the following formula, that is identical to the one that holds in a tangential quadrilateral (see [5, p.122]).

Corollary 4.1. *The tangency chords in an extangential quadrilateral with consecutive sides a, b, c, d satisfy*

$$\left(\frac{k}{l}\right)^2 = \frac{bd}{ac}.$$

Proof. From Theorem 4.1 we get after simplification that

$$\frac{k}{l} = \sqrt{\frac{(e-h)(f-g)}{(e-f)(h-g)}} = \sqrt{\frac{db}{ac}}$$

and the result follows. □

To prove the next formula we need the following simple angle equality.

Lemma 4.1. *If the extensions of opposite sides in a convex quadrilateral $ABCD$ intersect at E and F, then*

$$\angle C = \angle A + \angle E + \angle F$$

assuming that $\angle C > \angle A$.

Proof. In triangles ABF and ADE, we have that $B = \pi - A - F$ and $D = \pi - A - E$ (see Figure 3). Using the sum of angles in $ABCD$ yields

$$2\pi = A + (\pi - A - F) + C + (\pi - A - E),$$

and the formula follows after simplification. □

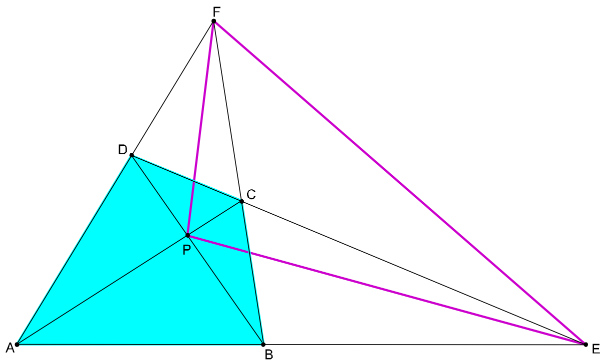

Figure 3. The diagonal point triangle EFP

Note that from a different perspective, this is a formula about the exterior angle C in a *concave* quadrilateral $AECF$, where the interior angle $ECF > \pi$.

Now we can derive a formula for the angle between the tangency chords in an extangential quadrilateral.

Theorem 4.2. *The acute angle φ between the tangency chords in an extangential quadrilateral with tangent lengths e, f, g, h is given by*

$$\cos\varphi = \frac{|eg - fh|}{\sqrt{(e-f)(f-g)(g-h)(h-e)}}.$$

Proof. In an extangential quadrilateral $ABCD$ with an excircle outside of C, let the excircle be tangent to the extensions of AB, BC, CD, DA at W, X, Y, Z respectively, and V be the intersection between the extensions of the tangency chords WY and ZX. If the extensions of opposite sides intersect at E and F, then we have that the exterior angles at E and F are $A+D$ and $A+B$ (see Figure 4), so $\angle VYC = \frac{\pi-A-D}{2}$ and $\angle VXC = \frac{\pi-A-B}{2}$ since triangles EWY and FXZ are isosceles due to the two tangent theorem.

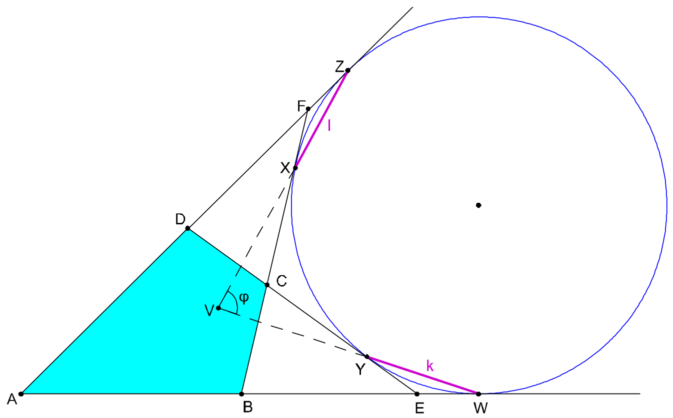

Figura 4. The tangency chords k, l and the angle φ between them

Applying Lemma 4.1 to the concave quadrilateral $VXCY$, we get

$$\begin{aligned}\varphi &= C-\frac{\pi-A-B}{2}-\frac{\pi-A-D}{2}\\ &= -\pi+\frac{A+B+C+D}{2}+\frac{A+C}{2}=\frac{A+C}{2}.\end{aligned} \tag{15}$$

Then from Theorem 3.2 (iv) and (i), we have for the area K of an extangential quadrilateral that

$$abcd-(eg-fh)^2=K^2=abcd\sin^2\frac{A+C}{2}=abcd\sin^2\varphi=abcd\big(1-\cos^2\varphi\big).$$

Hence

$$abcd\cos^2\varphi=(eg-fh)^2$$

and the formula follows since $a=e-f$, $b=f-g$, $c=h-g$ and $d=e-h$, where we reversed the order of the tangent lengths in the last two expressions to get a more symmetric formula. □

5. The area of the diagonal point triangle

In [8] and [10] we derived formulas for the area of the diagonal point triangle EFP in a cyclic and a tangential quadrilateral respectively, where E and F are the points where the extensions of opposite sides intersect and P is the intersection of the diagonals (see Figure 3).[3] Here we shall do the same for an extangential quadrilateral. As it turns out, the formula for the area of the diagonal point triangle belonging to an extangential quadrilateral has the exact same form as the corresponding formula that holds in a tangential quadrilateral.

Theorem 5.1. *If e, f, g, h are the tangent lengths in an extangential quadrilateral with no pair of opposite parallel sides, then the associated diagonal point triangle has the area*

$$T=\frac{2efghK}{|ef-gh||eh-fg|}$$

where

$$K=\sqrt{(e-f+g-h)(-efg+fgh-ghe+hef)}$$

is the area of the quadrilateral.

Proof. Again we will use that the only difference between a tangential quadrilateral and an extangential quadrilateral with tangent lengths e, f, g, h are the signs on f and h. Thus, since according to Theorem 1 in [10] the diagonal point triangle belonging to a tangential quadrilateral has the area

$$T=\frac{2efghK}{|ef-gh||eh-fg|},$$

then we get the corresponding area for an extangential quadrilateral by making the changes $f\to -f$ and $h\to -h$. But these changes do not effect the formula, since the extra minus signs cancel in pairs. □

[3] In those papers the notations on the vertices of the diagonal point triangle were slightly different.

Note that the area K in terms of the tangent lengths are different for a tangential and an extangential quadrilateral (see (8) and Theorem 3.1), so their diagonal point triangles does not have the same area.

6. Conditions for an extangential quadrilateral to be a kite

A *kite* is a quadrilateral that has two pairs of congruent adjacent sides.[4] Thus all kites have an excircle since their sides satisfy one of the conditions (1) or (2). Here we will address the possibility of the converse, that is, what additional property an extangential quadrilateral must have to be a kite? The corresponding investigation for tangential quadrilaterals was conducted by us in [7].

Theorem 6.1. *These statements are equivalent in an extangential quadrilateral:*
(i) *The quadrilateral is a convex kite.*
(ii) *The area is half the product of the diagonals.*
(iii) *The diagonals are perpendicular.*
(iv) *The tangency chords have equal length.*
(v) *One pair of opposite tangent lengths have equal length.*
(vi) *The bimedians have equal length.*[5]
(vii) *The product of opposite sides are equal.*

Proof. Let the extangential quadrilateral $ABCD$ have the consecutive sides a, b, c, d. We shall prove that each of the statements (i) through (vi) is equivalent to (vii); then all seven of them are equivalent.

(i) If the quadrilateral is a kite with $a = d$ and $b = c$, then $ac = bd$.

Conversely, if the quadrilateral is extangential ($a+b = c+d$ or $a+d = b+c$) and it holds that $ac = bd$, then assuming the first condition of the two holds (the proof in the second case is the same), we get

$$a + b = c + \frac{ac}{b} \quad \Leftrightarrow \quad ab - ac + b^2 - bc = 0 \quad \Leftrightarrow \quad (a + b)(b - c) = 0.$$

Hence $b = c$ and thus $a = d$, which proves that the quadrilateral is a kite.

(ii) Using Theorem 3.2 (ii) yields

$$K = \tfrac{1}{2}\sqrt{(pq)^2 - (ac - bd)^2} = \tfrac{1}{2}pq \quad \Leftrightarrow \quad ac = bd.$$

(iii) We use the well-known formula $K = \frac{1}{2}pq \sin\theta$ for the area of a convex quadrilateral (a proof was given in [3]), where θ is the angle between the diagonals p and q. From

$$K = \tfrac{1}{2}\sqrt{(pq)^2 - (ac - bd)^2} = \tfrac{1}{2}pq \sin\theta$$

we get

$$\theta = \frac{\pi}{2} \quad \Leftrightarrow \quad ac = bd.$$

(iv) Corollary 4.1 directly yields that $k = l$ is equivalent to $ac = bd$.

[4]We only consider convex kites (even though there are concave kites as well), since we will use a few formulas that hold in convex quadrilaterals.

[5]A bimedian in a quadrilateral is a line segment connecting the midpoints of two opposite sides.

(v) Let the tangent lengths be e, f, g, h, where $a = e - f$, $b = f - g$, $c = h - g$, and $d = e - h$. Then we have

$$\begin{aligned} & ac = bd \\ \Leftrightarrow \quad & (e-f)(h-g) = (f-g)(e-h) \\ \Leftrightarrow \quad & eh - ef + fg - gh = 0 \\ \Leftrightarrow \quad & (e-g)(h-f) = 0 \end{aligned}$$

which is true when (at least) one pair of opposite tangent lengths have equal length.

(vi) In the proof of Theorem 7 in [6] we noted that the length of the bimedians m and n in a convex quadrilateral are

$$m = \tfrac{1}{2}\sqrt{2(b^2+d^2)-4v^2}, \qquad n = \tfrac{1}{2}\sqrt{2(a^2+c^2)-4v^2}$$

where v is the distance between the midpoints of the diagonals. Thus we have

$$m = n \quad \Leftrightarrow \quad a^2+c^2 = b^2+d^2 \quad \Leftrightarrow \quad ac = bd$$

where the last equivalence is due to (9). □

7. The area of an exbicentric quadrilateral

A cyclic tangential quadrilateral is commonly known as a bicentric quadrilateral. A cyclic *extangential* quadrilateral was called an *ex-bicentric quadrilateral* in [13, p.44] (see Figure 5). We conclude this paper by discussing five formulas for the area of exbicentric quadrilaterals. Direct consequences of Theorems 3.2 (i) and 3.3 are the following inequalities.

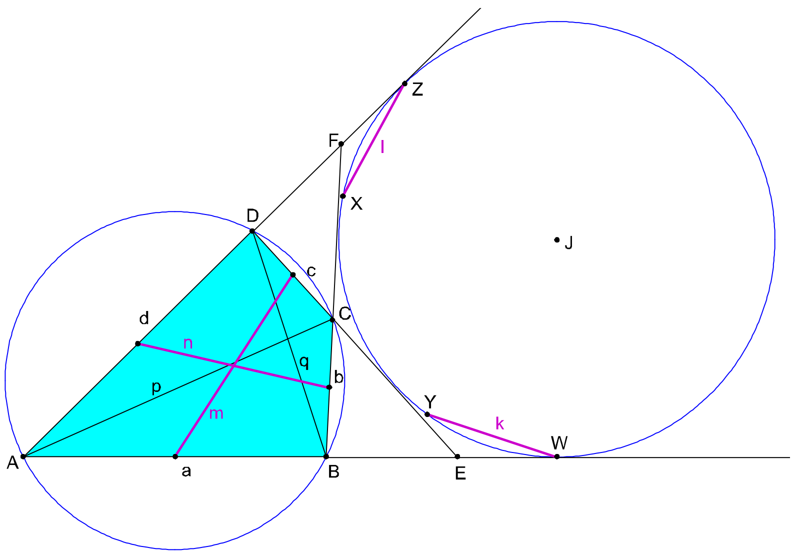

Figure 5. An exbicentric quadrilateral

Theorem 7.1. *The area of an extangential quadrilateral $ABCD$ with sides a, b, c, d and excenter J satisfies*
(i) $K \le \sqrt{abcd}$
(ii) $K \le |AJ \cdot CJ - BJ \cdot DJ|$
where equality holds in any of them if and only if the quadrilateral is also cyclic. However, in the second formula equality does not hold in rectangles.

Proof. (i) An extangential quadrilateral has the area

$$K = \sqrt{abcd}\sin\frac{A+C}{2} \le \sqrt{abcd}$$

with equality if and only if opposite angles are supplementary angles. That is a well-known characterization for a cyclic quadrilateral.

(ii) This proof is the same as (i), starting with Theorem 3.3, which is not valid for parallelograms. The exception for rectangles is due to the fact that a cyclic parallelogram is a rectangle, since the opposite angles are both supplementary and equal, so they are right angles. □

We note that $K = \sqrt{abcd}$ is the same formula that gives the area of a bicentric quadrilateral (see for instance [6, pp.155–156] for a short proof).

The area of an exbicentric quadrilateral can be expressed in terms of the tangent lengths $e = AW$, $f = BX$, $g = CY$, $h = DZ$ (see Figure 5) in a simpler way than in Theorem 3.1.

Theorem 7.2. *An exbicentric quadrilateral with tangent lengths e, f, g, h has the area*

$$K = |e - f + g - h|\sqrt[4]{efgh}$$

if it is not a rectangle.

Proof. Using that $eg = fh$ in Theorem 3.1 yields

$$\begin{aligned} K^2 &= (e - f + g - h)(-eg(f + h) + fh(g + e)) \\ &= (e - f + g - h)eg(e - f + g - h) = (e - f + g - h)^2\sqrt{efgh} \end{aligned}$$

and the formula follows. It is not valid in rectangles since they are the cyclic parallelograms. □

In [5, pp.129–130] and [6, p.161] we proved two beautiful formulas for the area of a bicentric quadrilateral in terms of the tangency chords and diagonals or bimedians. Since the derivations of these formulas only used the formulas in Corollary 4.1 and the equality case in Theorem 7.1 (i) (these two are valid in both bicentric and exbicentric quadrilaterals), Ptolemy's theorem (holds in cyclic quadrilaterals) and two area formulas for convex quadrilaterals, the results also hold in exbicentric quadrilaterals. Thus we will not repeat the derivations here.

Theorem 7.3. *The area of an exbicentric quadrilateral with tangency chords k, l, bimedians m, n, and diagonals p, q is given by the formulas*

$$K = \frac{klpq}{k^2 + l^2} \qquad \text{and} \qquad K = \left|\frac{m^2 - n^2}{k^2 - l^2}\right| kl$$

where the second does not apply to kites.

The distances k, l, m, n, p, q are shown in Figure 5. The second formula is not valid when $m = n$ or $k = l$. In Theorem 6.1 we proved that an extangential quadrilateral is a kite in those cases.

References

[1] *Art of Problem Solving*, China Team Selection Test **2003**, http://www.artofproblemsolving.com/Forum/resources.php?c=37&cid=47&year=2003

[2] Durell, C. V. and Robson, A., *Advanced Trigonometry*, G. Bell and Sons, London, **1930**. New edition by Dover Publications, Mineola, **2003**.

[3] Harries, J., *Area of a Quadrilateral*, Math. Gazette, **86 (2002)** 310–311.

[4] Josefsson, M., *On the inradius of a tangential quadrilateral*, Forum Geom., **10 (2010)** 127–34.

[5] Josefsson, M., *Calculations concerning the tangent lengths and tangency chords of a tangential quadrilateral*, Forum Geom., **10 (2010)** 119–130.

[6] Josefsson, M., *The area of a bicentric quadrilateral*, Forum Geom., **11 (2011)** 155–164.

[7] Josefsson, M., *When is a tangential quadrilateral a kite*?, Forum Geom., **11 (2011)** 165–174.

[8] Josefsson, M., The area of the diagonal point triangle, Forum Geom., **11 (2011)** 213–216.

[9] Josefsson, M., Similar metric characterizations of tangential and extangential quadrilaterals, Forum Geom., **12 (2012)** 63–77.

[10] Josefsson, M., The diagonal point triangle revisited, Forum Geom., **14 (2014)** 381–385.

[11] Josefsson, M., *More characterizations of extangential quadrilaterals*, International Journal of Geometry, **5 (2) (2016)** 62–76.

[12] Pech, P., *Selected topics in geometry with classical vs. computer proving*, World Scientific Publishing, **2007**.

[13] Radić, M., Kaliman, Z. and Kadum, V., *A condition that a tangential quadrilateral is also a chordal one*, Math. Commun., **12 (2007)** 33–52.

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