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THE NEW INEQUALITY IN A CYCLIC POLYGON

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Abstract. In this paper we give a proof of new inequality in arbitrary cyclic polygon.

1. Introduction

Let n points $A_1, A_2, ..., A_n$ lie on a circle, find structure geometry of $A_1, A_2, ..., A_n$ such that $\sum_{i<j} A_iA_j$ is maximum. The answer of this question in the statement of the theorem as followings:

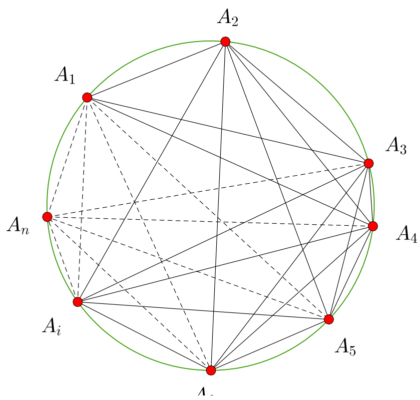

Figure 1

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Theorem 1.1. *Let n-regular polygon $X_1X_2\cdots X_n$ with the circumcribed circle (O). Let n points $A_1, A_2, \cdots, A_n$ lie on the circle (O) then:*

$$\sum_{i<j} A_iA_j \leq \sum_{i<j} X_iX_j$$

Equality holds if only if $A_1, A_2, ..., A_n$ are the vertices of a n regular polygon with the circumcribed circle (O).

We give the proof of theorem 1.1 in item 2 below:

2. PROOF OF THE ISOPERIMETRIC INEQUALITY

Proof. The case $n = 3$ is trivial and well-known [1].

Case 1: n odd, $n \geq 5$ denote $n = 2m+1, m \geq 2$.

We choose WLOG the circle $\omega : x^2+y^2 = 1$ and the points $A_i(\cos 2t_i, \sin 2t_i)$ for $i = 1, 2, \cdots, 2m+1$, suth that $0 = t_1 < t_2 < \cdots < t_{2m+1} < \pi$. From here we deduce that $A_iA_j = 2\sin(t_j - t_i)$ for $i = 1, 2, \cdots, 2m+1$. So we need to maximize the sum $\sum_{1\leq i<j\leq 2m+1} \sin(t_j - t_i)$. Since the function f: $(0, \pi) \to R$, $f(x) = \sin x$ is strictly concav, so by the Jensen's inequality we have:

$$\sum_{k=2}^{2m+1} \sin(t_k - t_{k-1}) \leq 2m \sin\left(\frac{1}{2m}\sum_{k=2}^{2m+1}(t_k - t_{k-1})\right) = 2m \sin\left(\frac{1}{2m}\cdot t_{2m+1}\right)$$

$$\sum_{k=2m+1}^{2m+1} \sin(t_k - t_{k-1}) = \sin(t_{2m+1})$$

$$\sum_{k=3}^{2m+1} \sin(t_k - t_{k-2}) \leq (2m-1)\sin\left(\frac{1}{2m-1}\sum_{k=3}^{2m+1}(t_k - t_{k-2})\right) =$$

$$= (2m-1)\sin\left(\frac{1}{2m-1}(t_{2m+1} + t_{2m} - t_2)\right)$$

$$\sum_{k=2m}^{2m+1} \sin(t_k - t_{k-2m+1}) \leq 2\sin\left(\frac{1}{2}(t_{2m+1} + t_{2m} - t_2)\right)$$

In general, if $2 \leq i \leq m+1$, then we have:

$$\sum_{k=i}^{2m+1} \sin(t_k - t_{k-i+1}) \leq (2m+2-i)\sin\left(\frac{1}{2m+2-i}\sum_{k=i}^{2m+1}(t_k - t_{k-i+1})\right) =$$

$$= (2m+2-i)\sin\left(\frac{1}{2m+2-i}(t_{2m+1} + t_{2m} + \cdots + t_{2m+3-i}) - (t_{i-1} + \cdots + t_1)\right)$$

$$\sum_{k=2m+3-i}^{2m+1} \sin(t_k - t_{k-2m-2+i}) \leq (i-1)\sin\left(\frac{1}{i-1}\sum_{k=i}^{2m+1}(t_k - t_{k-2m-2i+1})\right) =$$

$$= (i-1)\sin\left(\frac{1}{i-1}\left((t_{2m+1}+t_{2m}+\cdots+t_{2m+3-i})-(t_{i-1}+\cdots+t_1)\right)\right)$$

Also, let's observe that:

$$\sum_{i=2}^{m+1}\sum_{k=i}^{2m+1}\sin(t_k-t_{k-i+1})+\sum_{i=2}^{m+1}\sum_{k=2m+3-i}^{2m+1}\sin(t_k-t_{k-2m-2+i})=$$

$$=\sum_{1\le i<j\le 2m+1}\sin(t_j-t_i)$$

For every $i\in\{2,3,\cdots,m+1\}$ define the function $f_i:(0,(i-1)\pi)\to R$, as

$$f_i(x)=(2m+2-i)\sin\left(\frac{x}{2m+2-i}\right)+(i-1)\sin\left(\frac{x}{i-1}\right)$$

we have:

$$\begin{aligned} f_i'(x) &= \cos\left(\frac{x}{2m+2-i}\right)+\cos\left(\frac{x}{i-1}\right)\\ &= 2\cos\left(\frac{(2m+1)x}{2(2m+2-i)(i-1)}\right)\cos\left(\frac{(2m+3-2i)x}{2(2m+2-i)(i-1)}\right)\end{aligned}$$

But,

$$\frac{(2m+3-2i)x}{2(2m+2-i)(i-1)}<\frac{(2m+3-2i)(i-1)\pi}{2(2m+2-i)(i-1)}=\frac{(2m+3-2i)\pi}{2(2m+2-i)}<\frac{\pi}{2},$$

for $i\in\{2,3,\cdots,m+1\}$ and $x\in(0,(i-1)\pi)$ and

$$\frac{(2m+1)x}{2(2m+2-i)(i-1)}<\frac{(2m+1)\pi}{2(2m+2-i)}<\pi,$$

for $i\in\{2,3,\cdots,m+1\}$ and $x\in(0,(i-1)\pi)$.

In conclusion, $\cos\left(\frac{(2m+1)x}{2(2m+2-i)(i-1)}\right)>0$ with $\forall x\in(0,(i-1)\pi)$ and the function $\cos\left(\frac{(2m+1)x}{2(2m+2-i)(i-1)}\right)$ is strictly decreasing $on(0,(i-1)\pi)$. So that x_0 is critiacal point for f_i if only if $\frac{(2m+1)x}{2(2m+2-i)(i-1)}=\frac{\pi}{2}\Leftrightarrow x_0=\frac{(2m+2-i)(i-1)\pi}{(2m+1)}$. From the above considerations, $0<\frac{(2m+2-i)(i-1)\pi}{(2m+1)}<(i-1)\pi$. Thus,

$$\max f_i=f_i\left(\frac{(2m+2-i)(i-1)\pi)}{2m+1}\right)=(2m+1)\sin\left(\frac{(i-1)\pi}{2m+1}\right).$$

In conclusion,

$$\sum_{k=i}^{2m+1}\sin(t_k-t_{k-i+1})+\sum_{k=2m+3-i}^{2m+1}\sin(t_k-t_{k-2m-2+i})\le(2m+1)\sin\left(\frac{(i-1)\pi}{2m+1}\right).$$

Combining with the lemma, we deduce that equality holds if and only if

$$t_i-t_1=t_{i+1}-t_2=\cdots=t_{2m+1}-t_{2m+2-i}=\frac{(i-1)\pi}{2m+1}$$

and

$$t_{2m+3-i}-t_1=t_{2m+4-i}-t_2=\cdots=t_{2m+1}-t_{i-i}=\frac{(2m+2-i)\pi}{2m+1}.$$

Since $t_1=0$, then we deduce that $t_k=\frac{(k-1)\pi}{2m+1}$, for $k=1,2,\cdots,2m+1$.

Hence the maximum value is achieved iff the polygon is regular. And note that if the polygon is regular, then we deduce from above that:

$$\sum_{1\le i<j\le 2m+1} \sin(t_j - t_i) = (2m+1)\sum_{i=2}^{m+1} \sin\left(\frac{(i-1)\pi}{2m+1}\right).$$

In closed form:

$$\sum_{1\le i<j\le 2m+1} \sin(t_j - t_i) = (2m+1)\frac{\sin\frac{m\pi}{2(2m+1)}\sin\frac{(m+1)\pi}{2(2m+1)}}{\sin\frac{\pi}{2(2m+1)}}.$$

Case 2: n is even and $n \ge 4$. Analogously to case 1.

References

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