INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 5 (2016), No. 2, 5 - 21

Cyclic quadrilaterals corresponding to a given Varignon parallelogram

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Abstract. In this paper, we will study the cyclic quadrilaterals that have as a Varignon parallelogram any given parallelogram.

1. Introduction

The following result is well-known.

Theorem 1.1 (Varignon Theorem, 1731)**.** *Let $ABCD$ be a quadrilateral. If M, N, P, Q are the midpoints of the sides AB, BC, CD, and DA respectively, then $MNPQ$ is a parallelogram and $2T[MNPQ] = T[ABCD]$, where $T[ABCD]$ is the area of quadrilateral $ABCD$.*

In [4] one reciprocal theorem of Theorem 1.1 is demonstrated.

Theorem 1.2. *Given non collinear points so that $MNPQ$ is a parallelogram and considering an arbitrary point A in the plane of $MNPQ$, there exist B, C, D so that, M, N, P, Q are midpoints of sides AB, BC, CD, and DA respectively.*

In this paper, we will consider convex quadrilaterals. If $ABCD$ is a convex quadrilateral, M, N, P, Q are the midpoints of the sides AB, BC, CD and DA respectively, then the Varignon parallelogram corresponding to $ABCD$ quadrilateral is convex. The $MNPQ$ parallelogram, except for the points M, N, P, Q is situated in the interior of $ABCD$ quadrilateral.

According to Theorem 1.1, the quadrilateral $MNPQ$ is call the *Varignon's parallelogram* corresponding to $ABCD$ quadrilateral.

Theorem 1.2 implies that given $MNPQ$ parallelogram there is an infinite number of quadrilaterals that have as a Varignon parallelogram the $MNPQ$ parallelogram.

The following result is known (see [3]).

2010 *Mathematics Subject Classification.* 51M04, 51M15.
Key words and phrases. Cyclic quadrilaterals, Varignon parallelogram.
Received: 4.03.2015. In revised form: 4.04.2015. Accepted: 18.11.2015.

Theorem 1.3. *Let $ABCD$ be a cyclic quadrilateral, ω the center of the circumcircle of $ABCD$. If $MNPQ$ is the Varignon's parallelogram corresponding to $ABCD$ quadrilateral, then $\omega M \perp AB$, $\omega N \perp BC$, $\omega P \perp CD$ and $\omega Q \perp DA$.*

In this paper, we will solve the following problem: given the $MNPQ$ parallelogram, we will determine the geometrical locus of the points ω with the property that there exists a cyclic quadrilateral $ABCD$, the centre of the circumscribed circle of the $ABCD$ quadrilateral is ω and $MNPQ$ is the Varignon parallelogram corresponding to $ABCD$ quadrilateral.

2. Main results

Case I. Let $MNPQ$ be a parallelogram corresponding to the cyclic quadrilateral $ABCD$ and we suppose that ω, the centre of the circumscribed circle of the $ABCD$ quadrilateral, is situated in the interior of the $MNPQ$ parallelogram.

Lemma 2.1. *Let $ABCD$ be a cyclic quadrilateral, ω the centre of the circumscribed circle of the $ABCD$ quadrilateral, $MNPQ$ the corresponding Varignon parallelogram to the $ABCD$ quadrilateral, $M \in AB$, $N \in BC$, $P \in CD$ and $Q \in DA$. If ω is situated in the interior of the $MNPQ$ parallelogram, then $\widehat{\omega QM} \equiv \widehat{\omega NM}$ and the analogs.*

Proof. The quadrilaterals ωQAM and ωMBN are cyclic (Fig. 2.1), therefore $\widehat{\omega QM} \equiv \widehat{\omega AM}$ and $\widehat{\omega NM} \equiv \widehat{\omega BM}$ respectively. But the triangle ωAB is isosceles, therefore $\widehat{\omega AM} \equiv \widehat{\omega BM}$ and according to all the congruences above, yields the conclusion of the lemma. □

Next, we prove the existence of a point ω in an arbitrary parallelogram $MNPQ$ such that $\widehat{\omega QM} \equiv \widehat{\omega NM}$.

Proposition 2.1. *There exist a point ω situated in the interior of the parallelogram $MNPQ$ such that $\widehat{\omega QM} \equiv \widehat{\omega NM}$.*

Proof. We construct the straightline NT', where $T' \in (MQ)$ and $T'U \| MN$, $U \in (NP)$, implies $\widehat{T'NM} \equiv \widehat{NT'U}$. If $\{V\} = T'N \cap MU$ and let V' be the isogonal conjugate of a point V with respect to a triangle $MT'U$ is constructed by reflecting the line $T'V$ about the angle bisector of T', then $\widehat{MT'V'} \equiv \widehat{VT'U}$. Finally, we construct the parallel through Q to the line $T'V'$ which intersects the line $T'N$ in ω. Therefore, we have $\widehat{\omega QM} \equiv \widehat{\omega NM}$. Hence, for every straightline NT', with $T' \in (MQ)$, there is a single point ω such that $\widehat{\omega QM} \equiv \widehat{\omega NM}$. □

Lemma 2.2. *If ω is a point situated in the interior of the $MNPQ$ parallelogram and $\widehat{\omega QM} \equiv \widehat{\omega NM}$, then $\widehat{\omega MN} \equiv \widehat{\omega PN}$.*

Proof. We note $m(\widehat{\omega QM}) = \alpha$, $m(\widehat{QMN}) = a$, $m(\widehat{\omega MN}) = x$, $m(\widehat{\omega PN}) = y$, where $\alpha, a, x, y \in (0°, 180°)$. We have to prove that $x = y$ (Fig. 2.2).

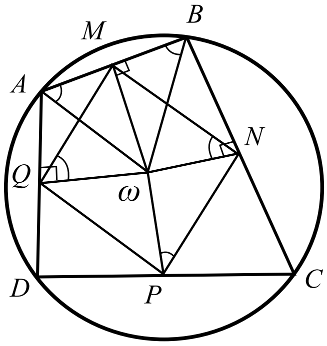

Fig. 2.1

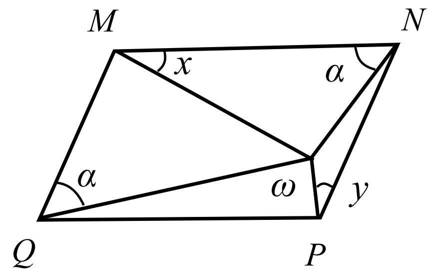

Fig. 2.2

In the triangles $MQ\omega$ and $MN\omega$, according to the sine theorem, yields $\frac{\sin\widehat{\omega QM}}{\omega M}=\frac{\sin\widehat{QM\omega}}{\omega Q}$ and $\frac{\sin\widehat{\omega NM}}{\omega M}=\frac{\sin\widehat{\omega MN}}{\omega N}$, or $\frac{\sin\alpha}{\omega M}=\frac{\sin(a-x)}{\omega Q}$ and $\frac{\sin\alpha}{\omega M}=\frac{\sin x}{\omega N}$, from where $\frac{\sin(a-x)}{\sin x}=\frac{\omega Q}{\omega N}$, equivalent to

$$\sin a\operatorname{ctg}x-\cos a=\frac{\omega Q}{\omega N}. \tag{2.1}$$

Taking that $m(\widehat{\omega QP})=m(\widehat{\omega NP})=180^\circ-a-\alpha$ into account, analogously we obtain that

$$\sin a\operatorname{ctg}y-\cos a=\frac{\omega Q}{\omega N}. \tag{2.2}$$

From (2.1) and (2.2) yields that $\sin a\cdot\operatorname{ctg}x-\cos a=\sin a\cdot\operatorname{ctg}y-\cos a$, equivalent to $\operatorname{ctg}x=\operatorname{ctg}y$. Because $x,y\in(0,180^\circ)$, according to the previous equality, we obtain that $x=y$. □

Theorem 2.1. *Let ω be a point situated in the interior of $MNPQ$ parallelogram so that $\widehat{\omega QM}\equiv\widehat{\omega NM}$. If $AB\perp\omega M$, $BC\perp\omega N$, $CD\perp\omega P$ and $DA\perp\omega Q$, then $ABCD$ is a cyclic quadrilateral and ω is the center of the circumscribed circle of the $ABCD$ quadrilateral.*

Proof. Because $\widehat{\omega QM}\equiv\widehat{\omega NM}$, according to Lemma 2.2 yields $\widehat{\omega MN}\equiv\widehat{\omega PN}$. But $MNPQ$ is a parallelogram, which means that $\widehat{\omega QP}\equiv\widehat{\omega NP}$. From $AB\perp\omega M$, $DA\perp\omega Q$ and $BC\perp\omega N$ (Fig. 2.3), it results that the quadrilaterals ωQAM and ωMBN are cyclic, which means that $\widehat{\omega QM}\equiv\widehat{\omega AM}$ and $\widehat{\omega NM}\equiv\widehat{\omega BM}$. But $\widehat{\omega QM}\equiv\widehat{\omega NM}$ and taken all the above into consideration, yields $\widehat{\omega AM}\equiv\widehat{\omega BM}$. In conclusion, the triangle ωAB is isosceles, therefore

$$\omega A\equiv\omega B. \tag{2.3}$$

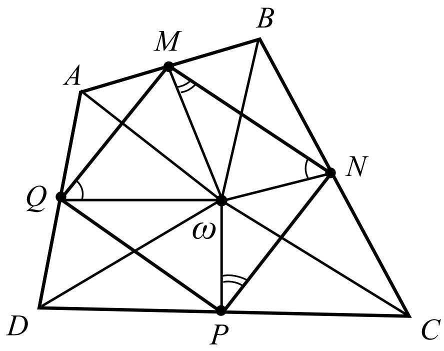

Fig. 2.3

Analogously, from $\widehat{\omega MN} \equiv \widehat{\omega PN}$ and $\widehat{\omega QP} \equiv \widehat{QNP}$, it results that $\omega B \equiv \omega C$, $\omega D \equiv \omega C$ respectively. Taking (2.3) into account, we obtain that $\omega A \equiv \omega B \equiv \omega C \equiv \omega D$, therefore $ABCD$ is a cyclic quadrilateral and ω is the centre of the circumscribed circle of the $ABCD$ quadrilateral. □

Remark 2.1. Theorem 2.1 is a reciprocal results to Lemma 2.1.

Let $MNPQ$ be a Varignon's parallelogram corresponding to the cyclic quadrilateral $ABCD$. Let ω be the centre of the circumcircle of $ABCD$. We suppose that ω is situated in the interior of the $MNPQ$ parallelogram. We will determine the plane area in which ω is situated.
Taking Theorem 1.3 and Lemma 2.1 into account, we have that $\omega M \perp AB$, $\omega N \perp BC$, $\omega P \perp CD$, $\omega Q \perp DA$ and $[ABCD] \cap [MNPQ] = [M, N, P, Q]$ (see Fig. 2.1), where $[ABCD]$ is the surface determined by the $ABCD$ quadrilateral and its interior.

If $m(\widehat{MNP}) < 90°$, then any perpendiculars in N on ωN does not intersect the interior of $MNPQ$ parallelogram (Fig. 2.4). If $m(\widehat{MNP}) \geq 90°$, we consider the following lines $d_1 \perp MN$, $d_2 \perp MN$, $M \in d_1$, $P \in d_2$, $d_3 \perp MQ$, $d_4 \perp MQ$, $M \in d_3$, $P \in d_4$, $d_1 \cap d_4 = \{S\}$, $d_3 \cap d_2 = \{T\}$.
Let $(d_1N$ be the open half plane determinated by d_1 line and the N point. Because ω point is situated in the interior of the $MNPQ$ parallelogram, $\omega M \perp AB$ and $MNPQ$ parallelogram, except for the points M, N, P, Q is situated in the interior of $ABCD$ quadrilateral, it results that $\omega \in (d_1N \cap (d_3Q$. Similarly $\omega \in (d_2Q \cap (d_4N$. The surface we are searching for is represented by the interior of the $MSPT$ parallelogram (Fig. 2.4).

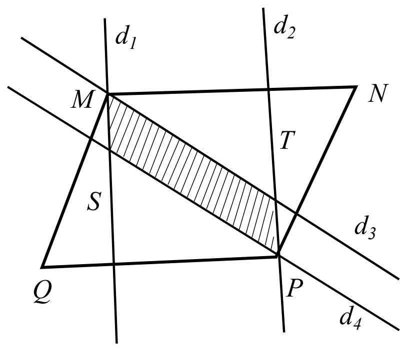

Fig. 2.4

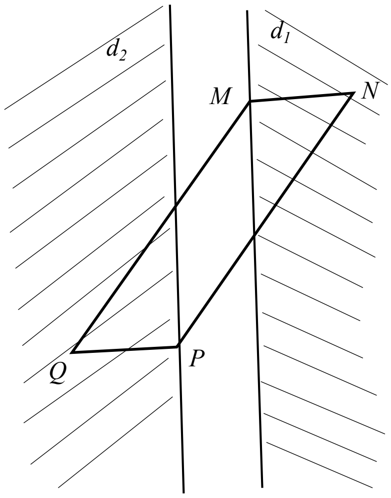

Fig. 2.5

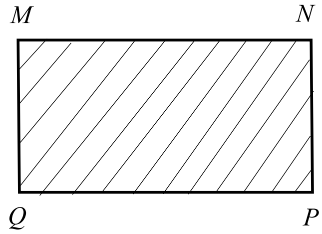

Fig. 2.6

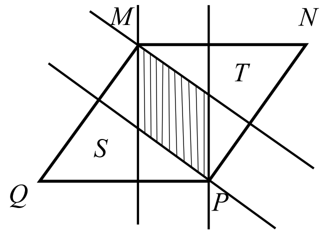

Fig. 2.7

Taking the previous remarks into account, we have a solution if and only if d_1 line intersects $[QP)$. We do not have a solution in a contrary case, then $c \geq 0$ (see Fig. 2.5).

If $MNPQ$ is a rectangle, then its interior is convenient for ω point (Fig. 2.6) and if $MNPQ$ is a rhomb, then the interior of the marked area from Fig. 2.7 is convenient.

In the following, let $MNPQ$ be a parallelogram, where its centre is the origin of the axis system (Fig. 2.8), $a > 0$, $b > 0$ and $c < a$.

Lemma 2.3. *Let $MNPQ$ be a given parallelogram* (see Fig. 2.8). *Then*

a) $m(\widehat{QMN}) \geq 90° \Leftrightarrow -c \leq a$.

b) If $MM' \perp QP$, $M' \in QP$, then $M' \in [QP) \Leftrightarrow -c \leq a$ and $c < 0$.

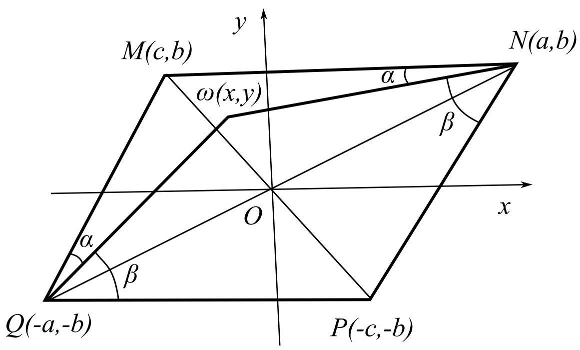

Fig. 2.8

Proof. a) We have that $MN = a - c$, $MQ = \sqrt{(a+c)^2 + 4b^2}$, $NQ = \sqrt{4a^2 + 4b^2}$ and $\cos\widehat{QMN} = \dfrac{MQ^2 + MN^2 - QN^2}{2MQ \cdot MN} = \dfrac{c^2 - a^2}{MQ \cdot MN}$. Then $m(\widehat{QMN}) \geq 90°$ if and only if $\cos\widehat{QMN} \leq 0$, equivalent to $c^2 - a^2 \leq 0$, equivalent to $(c-a)(c+a) \leq 0$, which is equivalent to $c + a \geq 0$, that yields a).

b) The point M' has the coordinates $M'(c, y_{M'})$ and $M' \in [QP)$ if and only if $-a \leq c < -c$, which yields b). $\square$

Theorem 2.2. *Let $MNPQ$ be a given parallelogram (Fig. 2.8). The geometric locus of ω points situated in the interior of the $MNPQ$ parallelogram so that $\widehat{\omega QM} \equiv \widehat{\omega NM}$ is:*

a) if $c \neq -\dfrac{b^2}{a}$, the hyperbola

$$(H) \quad bx^2 - by^2 - (a+c)xy + b^3 + abc = 0 \tag{2.4}$$

intersected with the interior of the $MNPQ$ parallelogram;

b) if $c = -\dfrac{b^2}{a}$, the lines

$$d' : y = \frac{b}{a}x \quad \text{and} \quad d'' : y = -\frac{a}{b}x \tag{2.5}$$

intersected with the interior of the $MNPQ$ parallelogram. In this case, $MNPQ$ becomes a rhomb.

Proof. Let ω be a point with the coordinates $\omega(x, y)$, we note $\alpha = m(\widehat{\omega QM}) = m(\widehat{\omega NM})$, $\beta = m(\widehat{\omega QP}) = m(\widehat{\omega NP})$ and by $m_{\omega N}$ the slope of the ωN line. Then

$$m_{\omega N} = \operatorname{tg}\alpha = \frac{b-y}{a-x}, \tag{2.6}$$

$m_{\omega Q} = \operatorname{tg}\beta = \dfrac{y+b}{x+a}$ and $m_{MQ} = \operatorname{tg}(\alpha+\beta) = \dfrac{2b}{c+a}$. Taking the last two equalities into account, yield

$$\operatorname{tg}\alpha = tg((\alpha+\beta)-\beta) = \frac{\operatorname{tg}(\alpha+\beta) - \operatorname{tg}\beta}{1 + \operatorname{tg}(\alpha+\beta)\operatorname{tg}\beta} = \frac{\dfrac{2b}{c+a} - \dfrac{y+b}{x+a}}{1 + \dfrac{2b}{c+a}\cdot\dfrac{y+b}{x+a}},$$

from where

$$\operatorname{tg}\alpha = \frac{2bx - cy - bc - ay + ab}{a^2 + ac + ax + cx + 2by + 2b^2}. \tag{2.7}$$

From (2.6) and (2.7), after calculus yield (2.4).
If $a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$ is the general equation of the conic, then $\delta = a_{11}a_{22} - a_{12}^2 = -b^2 - \left(\dfrac{a+c}{2}\right)^2 < 0$ because $b > 0$ and

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \begin{vmatrix} b & -\dfrac{a+c}{2} & 0 \\ -\dfrac{a+c}{2} & -b & 0 \\ 0 & 0 & b^3 + abc \end{vmatrix}$$
$$= -b(b^2 + ac)\left(b^2 + \left(\frac{a+c}{2}\right)^2\right).$$

Because $a > 0$, $b > 0$, then $D = 0$ if and only if $b^2 + ac = 0$, which is equivalent to $c = -\dfrac{b^2}{a}$. So, if $c \neq -\dfrac{b^2}{a}$ it results that $\Delta \neq 0$ and (2.4) is a hyperbola. If $c = -\dfrac{b^2}{a}$ then (2.4) becomes $bx^2 - by^2 - \left(a - \dfrac{b^2}{a}\right)xy = 0$, which is equivalent to $abx^2 - (a^2 - b^2)xy - aby^2 = 0$, equivalent to $(ax + by)(bx - ay) = 0$, which yields (2.5). In this case, where $c = -\dfrac{b^2}{a}$, we have that $MN \equiv MQ \equiv \dfrac{a^2 + b^2}{a}$. Therefore the $MNPQ$ parallelogram becomes a rhomb. □

Corollary 2.1. *In the conditions of Theorem* 2.2*, if* $c \neq -\dfrac{b^2}{a}$*, then the hyperbola* (H) *defined by* (2.4) *has the property that its center is* $O(0,0)$*.*

Proof. If the equation of the hyperbola (H) is $f(x,y) = bx^2 - by^2 - (a+c)xy + b^3 + abc = 0$, then its center can be determined by solving the following system $\begin{cases} f'_x(x,y) = 0 \\ f'_y(x,y) = 0. \end{cases}$ We have $\begin{cases} 2bx - (a+c)y = 0 \\ -2by - (a+c)x = 0 \end{cases}$, from where we obtain the solution $\begin{cases} x = 0 \\ y = 0 \end{cases}$, which means that the center of the hyperbola (H) is $O(0,0)$. □

Remark 2.2. It can be easily checked that the vertices of the $MNPQ$ parallelogram belong to the hyperbola given by (2.4). The points N and Q are situated on the line given by $y = \dfrac{b}{a}x$. If $c = -\dfrac{b^2}{a}$, then the points M and P are situated on the line given by $y = -\dfrac{a}{b}x$.

Remark 2.3. The point S is situated at the intersection of lines MS of equation $x = c$ and PS of equation $y = -\dfrac{a+c}{2b}x + \dfrac{(a+c)c}{2b} - b$, so S has the coordinates $S\Big(c, -\dfrac{ac + c^2 + b^2}{b}\Big)$. Analogously, the point T has the coordinates
$T\Big(-c, \dfrac{ac + c^2 + b^2}{b}\Big)$. It is easily verified that the points S and T are situated on the hyperbola (H) given by (2.4) if $c \neq -\dfrac{b^2}{a}$, and are situated on the line d' given by (2.5) if $c = -\dfrac{b^2}{a}$.

Theorem 2.3. *Let* $MNPQ$ *be a given parallelogram,* $M(c,b)$*,* $N(a,b)$*,* $P(-c,-b)$*,* $Q(-a,-b)$*,* $a > 0$*,* $b > 0$*,* $c < a$*,* $-c \leq a$*,* $MS \perp QP$*,* $PT \perp MN$*,* $MT \perp PN$*,* $PS \perp MQ$*,* ω *a point situated in the interior of the* $MNPQ$ *parallelogram, so that* $\omega M \perp AB$*,* $\omega N \perp BC$*,* $\omega P \perp CD$ *and* $\omega Q \perp DA$*.*

(*i*) *If* $c < 0$*, then the* $ABCD$ *quadrilateral is cyclic and* ω *is the center of the circumscribed circle of the* $ABCD$ *quadrilateral if and only if:*

a) *if* $c \neq -\dfrac{b^2}{a}$*,* ω *belongs to the intersection between the hyperbola* (H) *determined by* (2.4) *and the interior of* $MSPT$ *parallelogram;*

b) *if* $c = -\dfrac{b^2}{a}$*,* ω *belongs to* $(MP) \cup [TS]$*.*

(ii) If $c \geq 0$, then there are not any ω points in the interior of $MSPT$ parallelogram.

Proof. In Case I, taking Lemma 2.1, Theorem 2.1, Lemma 2.3, Remark 2.3 and Theorem 2.2 into account, yields the demonstration. □

Case II. Let $ABCD$ be a cyclic quadrilateral and $MNPQ$ the corresponding Varignon parallelogram. We will study if ω, the center of the circumscribed circle of the $ABCD$ quadrilateral, can be situated on a side of $MNPQ$ parallelogram.

For example, if ω is situated in the interior of the PN side (Fig. 2.9), then $\omega N \perp BC$ and $\omega P \perp DC$, which is a contradiction. Therefore ω cannot be situated on the open sides of $MNPQ$ parallelogram.

We will study if ω can be situated in on one of the vertices of the $MNPQ$ parallelogram, for instance P (Fig. 2.10) and we note $AC \cap BD = \{S\}$, $AC \cap PN = \{T\}$, $BD \cap PQ = \{V\}$.

We have that PN, PQ are median lines in BDC triangle and ADC respectively, which yield $PN \| BD$ and $PQ \| AC$, from where $PTSV$ is parallelogram, so $\widehat{QPN} \equiv \widehat{DSC}$. But $m(\widehat{DSC}) = \dfrac{m(\widehat{AB}) + m(\widehat{CD})}{2}$ and since $m(\widehat{CD}) = 180°$, yields $m(\widehat{QPN}) > 90°$, which means $\widehat{QPN}$ is an obtuse. Therefore, the center of the circumscribed circle of the $ABCD$ quadrilateral can only be situated in a vertex of the Varignon parallelogram, if the angle corresponding to this vertex is obtuse angle. The side of $ABCD$ quadrilateral corresponding to this vertex is diameter of the circumscribed circle of the $ABCD$ quadrilateral (Fig. 2.10).

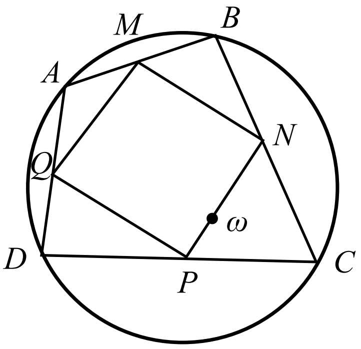

Fig. 2.9

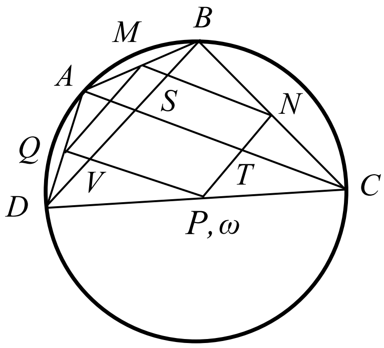

Fig. 2.10

Similar remarks from the Case I, if $d_1 \perp MN$, $M \in d_1$, we have a solution if and only if $d_1 \cap [QP \neq \emptyset$. Taking Lemma 2.3 into account, we have a solution if and only if $c < 0$ (see Fig. 2.8).

Lemma 2.4. *Let $MNPQ$ be a parallelogram, $m(\widehat{QPN}) > 90°$, $PM \perp AB$, $PN \perp BC$, $PQ \perp DA$, $m(\widehat{QMP}) + m(\widehat{QPD}) = 90°$, $Q \in (AD)$. If D, P, C are collinear, then the $ABCD$ quadrilateral is cyclic and the center of the circumscribed circle of $ABCD$ is P.*

Proof. The $PQAM$ and $PMBN$ quadrilaterals are cyclic (Fig. 2.11), that yields

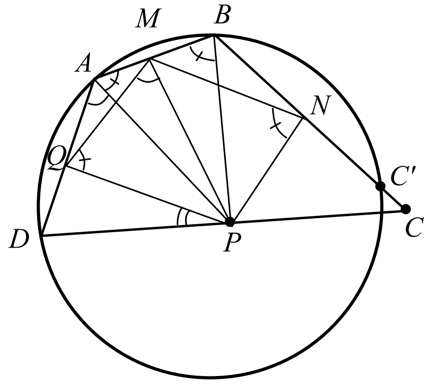

Fig. 2.11

(2.8) $\widehat{PAQ} \equiv \widehat{PMQ}$,

(2.9) $\widehat{PQM} \equiv \widehat{PAM}$

and respectively

(2.10) $\widehat{PNM} \equiv \widehat{PBM}$.

But $MNPQ$ is parallelogram, therefore $\widehat{PQM} \equiv \widehat{PNM}$ and taking (2.9) and (2.10) into account yields $\widehat{PAM} \equiv \widehat{PBM}$, which means that the PAB triangle is isosceles, from where

(2.11) $$PA \equiv PB.$$

In the DPQ triangle, $m(\widehat{QDP}) + m(\widehat{QPD}) = 90°$ and taking the hypothesis into account, yields

(2.12) $$\widehat{PMQ} \equiv \widehat{QDP}.$$

From (2.8) and (2.12), it results that $\widehat{PAQ} \equiv \widehat{QDP}$, so the triangle PAD is isosceles, from where

(2.13) $$PA \equiv PD.$$

From (2.11) and (2.13) it results that the points D, A, B are situated on a circle $\mathcal{C}$ of center P and radius PA. Let $\mathcal{C} \cap BC = \{B, C'\}$ and because $A, D,\ B, C' \in \mathcal{C}$ and $PQ \perp DA$, $PM \perp AB$, $PN \perp BC$, yields that the points A and D are symmetrical to Q, A and B are symmetrical to M and B and C' are symmetrical to N. According to Theorem 1.2, yields points D, P, C' are collinear and symmetrical to P. But $C, C' \in BC$, $C, C' \in DP$, the fact that the points D, P, C' and D, P, C are collinear, means that C and C' are coincident points. □

Remark 2.4. In Lemma 2.4 we have proved that for a $MNPQ$ parallelogram with $m(\widehat{QPN}) > 90°$, there is a cyclic quadrilateral $ABCD$, uniquely determinated, so that P is the center of the circumscribed circle of the $ABCD$ quadrilateral, and $MNPQ$ is the Varignon parallelogram corresponding to the $ABCD$ quadrilateral. Analogously, the point M has got the same property.

Theorem 2.4. *Let $MNPQ$ be a given parallelogram, $M(c, b)$, $N(a, b)$, $P(-c, -b)$, $Q(-a, -b)$, $a > 0$, $b > 0$, $c < a$, $-c \leq a$.*

(i) If $c < 0$, then there exists an unique cyclic quadrilateral $ABCD$ so that P is the center of the circumscribed circle of the $ABCD$ quadrilateral and $MNPQ$ is the Varignon parallelogram corresponding to the $ABCD$ quadrilateral. The point M has got the same property and the points P and M are situated on the (H) hyperbola determined by (2.4) if $c \neq -\dfrac{b^2}{a}$ and $P, M \in d''$ if $c = -\dfrac{b^2}{a}$.

(ii) If $c \geq 0$, then the points P, and respectively M, cannot be the center of the circumscribed circle of $ABCD$ quadrilateral, which means that $MNPQ$ is Varignon parallelogram corresponding to the $ABCD$ quadrilateral.

Proof. Taking Lemma 2.4 and remarks above into account, yields the demonstration. □

Case III. Let $ABCD$ be a cyclic quadrilateral and $MNPQ$ the Varignon parallelogram corresponding to the $ABCD$ quadrilateral. We will study if ω, the center of the circumscribed circle of the $ABCD$ quadrilateral can be situated in the exterior of the $MNPQ$ parallelogram.

Let ω be a point situated in the exterior of the parallelogram $MNPQ$ and $AC \cap BD = \{S\}$, $AC \cap PN = \{T\}$, $BD \cap PQ = \{V\}$ (Fig. 2.12). Because $\omega P \perp DC$ and ω is situated in the exterior of the $MNPQ$ parallelogram, it results that ω is situated in the exterior of the $ABCD$ quadrilateral. Because $PTSV$ is a parallelogram, we have that $\widehat{QPN} \equiv \widehat{DSC}$. But $m(\widehat{DSC}) = \dfrac{m(\widehat{AB}) + m(\widehat{CD})}{2}$ and since $m(\widehat{CD}) > 180°$, yields $m(\widehat{QPN} > 90°$, which means $\widehat{QPN}$ is obtuse angle.

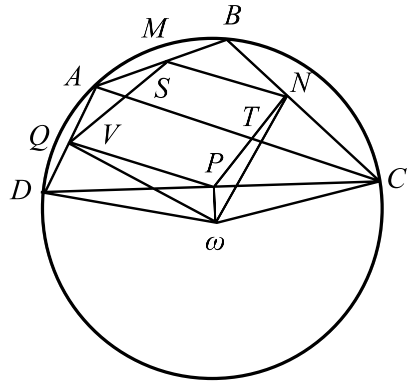

Fig. 2.12

Lemma 2.5. *Let $ABCD$ be a cyclic quadrilateral, ω the center of the circumscribed circle of the $ABCD$ quadrilateral, $MNPQ$ the Varignon parallelogram corresponding to the $ABCD$ quadrilateral, $M \in AB$, $N \in BC$, $P \in CD$ and $Q \in DA$. If ω is situated in the exterior of the $MNPQ$ parallelogram and $m(\widehat{QPN}) > 90°$, then*

a) $\widehat{\omega QP} \equiv \widehat{\omega NP}$

and

b) $\omega Q \cap Int\, MNPQ = \omega N \cap Int\, MNPQ = \emptyset$.

Proof. Because $\omega P \perp DC$, $\omega Q \perp AD$, $\omega N \perp BC$ we conclude that the ωDQP and the ωCNP quadrilaterals are cyclic, from where $\widehat{\omega DP} \equiv \widehat{\omega QP}$ and $\widehat{\omega CP} \equiv \widehat{\omega NP}$ (Fig. 2.12). But the ωDC triangle is isosceles, therefore $\widehat{\omega DP} \equiv \widehat{\omega CP}$ and taking the previous relations into account, yield part a) from this lemma.

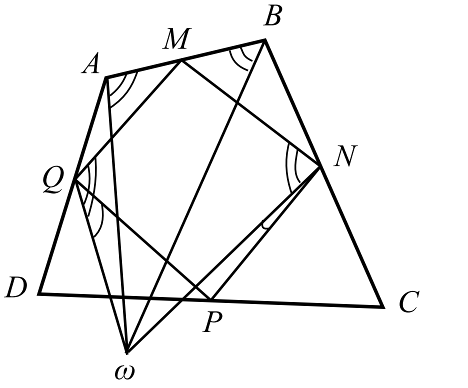

Fig. 2.13

If $\omega Q \cap Int\, MNPQ \neq \emptyset$ and $\omega N \cap Int\, MNPQ \neq \emptyset$, then $\omega \in Int\, MNPQ$ which is a contradiction.

Let $\omega Q \cap Int\, MNPQ = \emptyset$ and $\omega N \cap Int\, MNPQ \neq \emptyset$ (Fig. 2.13). Taking a) into account, yields $\widehat{\omega QP} \equiv \widehat{\omega NP}$, from where

(2.14)
$$m(\widehat{\omega NM}) < n(\widehat{PNM})$$

and

(2.15)
$$m(\widehat{\omega QM}) > m(\widehat{PQM}).$$

The ωNBM and the ωQAM quadrilaterals are cyclic, therefore $\widehat{\omega NM} \equiv \widehat{\omega BM}$ and $\widehat{\omega QM} \equiv \widehat{\omega AM}$. But the ωAB triangle is isosceles, so $\widehat{\omega AM} \equiv \widehat{\omega BM}$ and therefore we obtain

$$\widehat{\omega NM} \equiv \widehat{\omega QM}. \tag{2.16}$$

From (2.14)-(2.16) yield $m(\widehat{PQM}) < \omega(\widehat{PNM})$, which is a contradiction because $m(\widehat{PQM}) = m(\widehat{PNM})$. In conclusion, part b) takes place. □

Lemma 2.6. *Let $MNPQ$ be a parallelogram where $m(\widehat{QPN}) > 90°$, and ω is a point so that $\omega Q \cap Int\, MNPQ = \omega N \cap Int\, MNPQ = \emptyset$ and $\widehat{\omega QP} \equiv \widehat{\omega NP}$. Therefore $m(\widehat{\omega MQ}) + m(\widehat{\omega PQ}) = 180°$.*

Proof. In the triangle ωQP and ωNP, according to the law of sines (Fig. 2.12), we obtain $\dfrac{\sin\alpha}{\omega P} = \dfrac{\sin x}{\omega Q}$ and $\dfrac{\sin\alpha}{\omega P} = \dfrac{\sin(360° - b - x)}{\omega N}$, where we note $m(\widehat{\omega QP}) = \alpha$, $m(\widehat{QPN}) = b$ and $m(\widehat{\omega PQ}) = x$. From the relations above, yield

$$\frac{-\sin(b + x)}{\sin x} = \frac{\omega N}{\omega Q}. \tag{2.17}$$

In the triangles ωQP and ωNP, according to the law of sines, we obtain $\dfrac{\sin y}{\omega Q} = \dfrac{\sin(\alpha + a)}{\omega M}$ and $\dfrac{\sin(b - y)}{\omega N} = \dfrac{\sin(\alpha + a)}{\omega M}$, where $m(\widehat{\omega MQ}) = y$ and $m(\widehat{PQM}) = a$. From the last previous equalities, we obtain that

$$\frac{\sin(b - y)}{\sin y} = \frac{\omega N}{\omega Q}. \tag{2.18}$$

From (2.17) and (2.18), we have that $\dfrac{-\sin(b + x)}{\sin x} = \dfrac{\sin(b - y)}{\sin y}$, equivalent to $-\sin y \sin b \cos x - \sin y \sin x \cos b = \sin x \sin b \cos y - \sin x \sin y \cos b$, equivalent to $\sin b \sin(x + y) = 0$. Because $b \in (0°, 180°)$, so $\sin b \neq 0$, it results that $\sin(x + y) = 0$, from where $x + y = 180°$, which needs to be proved. □

Theorem 2.5. *Let ω be a point situated in the exterior of the parallelogram $MNPQ$, so that $\omega Q \cap Int\, MNPQ = \omega N \cap Int\, MNPQ = \emptyset$ and $\widehat{\omega QP} = \widehat{\omega NP}$. If $m(\widehat{QPN}) > 90°$, $AB \perp \omega M$, $BC \perp \omega N$, $CD \perp \omega P$ and $DS \perp \omega Q$, then $ABCD$ is a cyclic quadrilateral and ω is the center of the circumscribed circle of the $ABCD$ quadrilateral.*

Proof. From $AB \perp \omega M$ and $DA \perp \omega Q$, we obtain that the ωMAQ quadrilateral is cyclic (Fig. 2.14), from where

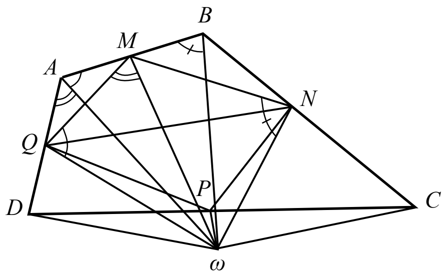

Fig. 2.14

$$\widehat{\omega AM} \equiv \widehat{\omega QM} \tag{2.19}$$

and

$$\widehat{\omega MQ} \equiv \widehat{\omega AQ}. \tag{2.20}$$

The quadrilateral ωMBN is cyclic, therefore

$$\widehat{\omega BM} \equiv \widehat{\omega NM}. \tag{2.21}$$

Because $MNPQ$ is a parallelogram, so $\widehat{PQM} \equiv \widehat{PNM}$ and from the hypothesis we have $\widehat{\omega QP} \equiv \widehat{\omega NP}$, therefore $\widehat{\omega QM} \equiv \widehat{\omega NM}$. Taking (2.19) and (2.21) into account, we obtain that $\widehat{\omega AM} \equiv \widehat{\omega BN}$, so the triangle ωAB is isosceles, so

$$\omega A \equiv \omega B. \tag{2.22}$$

The quadrilateral ωDQP is cyclic, which means that

$$m(\widehat{\omega DQ}) + m(\widehat{\omega PQ}) = 180°. \tag{2.23}$$

According to Lemma 2.6 we have that $m(\widehat{\omega MQ}) + m(\widehat{\omega PQ}) = 180°$ and taking (2.20) and (2.23) into account, yield $\widehat{\omega AQ} \equiv \widehat{\omega DQ}$, therefore the triangle ωAD is isosceles, so

$$\omega A \equiv \omega D. \tag{2.24}$$

The quadrilaterals ωDQP and ωCNP are cyclic, therefore $\widehat{\omega QP} \equiv \widehat{\omega DP}$ and $\widehat{\omega NP} \equiv \widehat{\omega CP}$. But, from the hypothesis we have $\widehat{\omega QP} \equiv \widehat{\omega NP}$, and then from the equalities above we deduce that $\widehat{\omega DP} \equiv \widehat{\omega CP}$. Therefore the triangle ωCD is isosceles, from where

$$\omega D \equiv \omega C. \tag{2.25}$$

From (2.16), (2.24) and (2.25), we have that the $ABCD$ quadrilateral is cyclic and ω is the center of the circumscribed circle of the $ABCD$ quadrilateral. □

Remark 2.5. In the previous conditions, if $\widehat{\omega QD} \equiv \widehat{\omega ND}$ then we obtain that $\widehat{\omega QM} \equiv \widehat{\omega NM}$, because $MNPQ$ is a parallelogram.

According to the ideas from Case I, we have the marked areas in Fig. 2.15 for $c < 0$ and in Fig. 2.16 for $c \geq 0$ respectively.

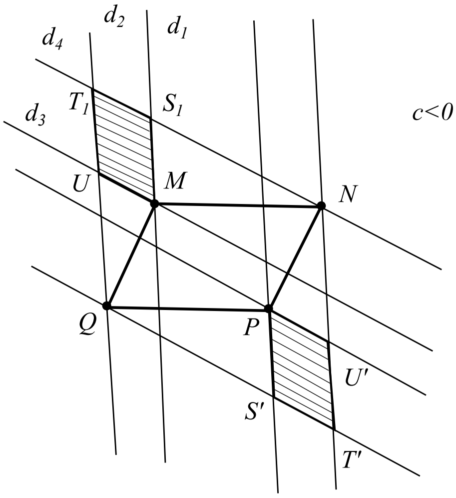

Fig. 2.15

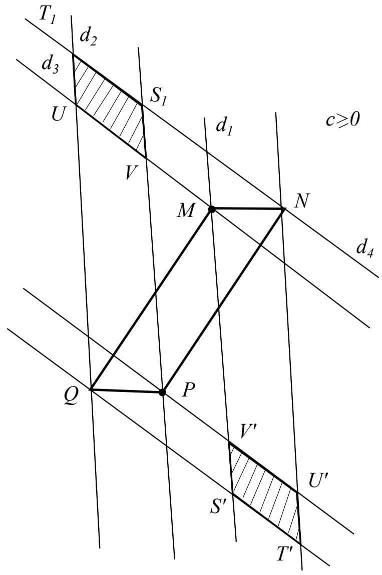

Fig. 2.16

In this case, let $MNPQ$ be a parallelogram and its center the origin of the axis system (Fig. 2.17), where $a > 0$, $b > 0$ and $c < a$.

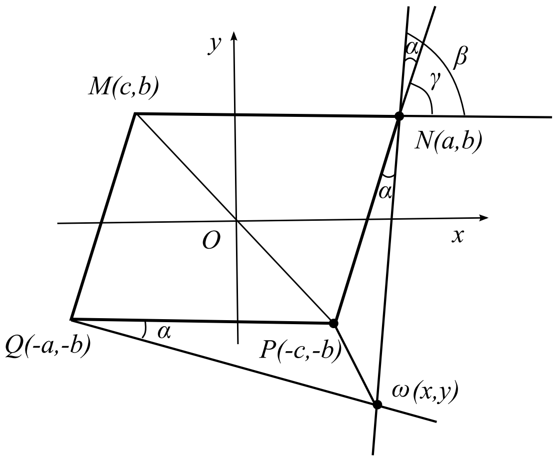

Fig. 2.17

Theorem 2.6. *Let $MNPQ$ be a given parallelogram with $m(\widehat{QPN}) > 90^\circ$* (Fig. 2.17). *The geometric locus of ω points situated in the exterior of the parallelogram $MNPQ$ so that $\omega Q \cap Int\, MNPQ = \omega N \cap Int\, MNPQ = \emptyset$ and $\widehat{\omega QP} \equiv \widehat{\omega NP}$ is:*

a) if $c \neq -\dfrac{b^2}{a}$, the hyperbola

$$(H) \quad bx^2 - by^2 - (a+c)xy + b^3 + abc = 0 \tag{2.26}$$

intersected with the exterior of the parallelogram $MNPQ$;

b) if $c = -\dfrac{b^2}{a}$, the lines

$$d' : y = \frac{b}{a}x \quad \text{and} \quad d'' : y = -\frac{a}{b}x \tag{2.27}$$

intersected with the exterior of the parallelogram $MNPQ$ (in this situation, $MNPQ$ becomes a rhomb).

Proof. Let $\omega(x,y)$ and we note $m(\widehat{\omega QP}) = m(\widehat{\omega NP}) = \alpha$, the measures of the angles formed by the lines ωN, PN with Ox axis by β, and γ respectively (Fig. 2.17).

We have that $m_{\omega Q} = \operatorname{tg}(180^\circ - \alpha) = \dfrac{y+b}{x+a}$, from where

$$\operatorname{tg}\alpha = -\frac{y+b}{x+a}\,. \tag{2.28}$$

On the other hand, $m_{PN} = \operatorname{tg}\gamma = \dfrac{2b}{a+c}$ and $m_{\omega N} = \operatorname{tg}\beta = \dfrac{y-b}{x-a}$. Then $\alpha = \beta - \gamma$, yields $\operatorname{tg}\alpha = \operatorname{tg}(\beta-\gamma)$, equivalent to

$$\operatorname{tg}\alpha = \frac{\operatorname{tg}\beta - \operatorname{tg}\gamma}{1+\operatorname{tg}\beta\operatorname{tg}\gamma} = \frac{\dfrac{y-b}{x-a} - \dfrac{2b}{a+c}}{1 + \dfrac{y-b}{x-a}\cdot\dfrac{2b}{a+c}}\,,$$

from where

$$\operatorname{tg}\alpha = \frac{ay - ab + cy - 2bx + bc}{ax - ac + cx - c^2 + 2by - 2b^2}\,. \tag{2.29}$$

From (2.28) and (2.29) after calculus, we obtain (2.26). □

Remark 2.6. The point T_1 is situated at the intersection of lines QU of equation $x = -a$ and S_1N of equation $y = -\dfrac{a+c}{2b}x + \dfrac{(a+c)a}{2b} + b$, so T_1 has the coordinates $T_1\Big(-a, \dfrac{ac+a^2+b^2}{b}\Big)$ and $T'\Big(a, -\dfrac{ac+a^2+b^2}{b}\Big)$, $V\Big(-c, \dfrac{ac+c^2+b^2}{b}\Big)$, $V'\Big(c, -\dfrac{ac+c^2+b^2}{b}\Big)$ analogously. It is easily verified that the points T_1, T', V, V' are situated on the hyperbola (H) given by (2.26) if $c \neq -\dfrac{b^2}{a}$. If $c = -\dfrac{b^2}{a}$, then the points $V, V' \in d'$ and $T_1, T' \in d''$.

In the following, see the ideas that led to the proof of Theorem 2.2.

Theorem 2.7. *Let $MNPQ$ be a given parallelogram, $M(c,b)$, $N(a,b)$, $P(-c,-b)$, $Q(-a,-b)$, $a>0$, $b>0$, $c<a$, $c<0$, $MS_1 \perp MN$, $T_1U \perp MN$, $MU \perp MQ$, $T_1S_1 \perp MQ$, $N \in T_1S_1$, $PS' \perp QP$, $U'T' \perp QP$, $PU' \perp PN$, $S'T' \perp PN$, $Q \in S'T'$* (Fig. 2.15), *ω a point situated in the exterior of the $MNPQ$ parallelogram so that $\omega M \perp AB$, $\omega N \perp BC$, $\omega P \perp CD$ and $\omega Q \perp DA$.*
The $ABCD$ quadrilateral is cyclic and ω is the center of the circumscribed circle of the quadrilateral $ABCD$ if and only if:

a) if $c \neq -\dfrac{b^2}{a}$, ω belongs to the intersection between the hyperbola (H) given by (2.26) *and the set $[MS_1T_1U] \cup [PS'T'U'] \setminus \{M,P\}$*(Fig. 2.15)*;*

b) if $c = -\dfrac{b^2}{a}$, ω belongs to set $[T_1M) \cup (PT']$ (Fig. 2.15).

Proof. Taking Lemma 2.5, Theorem 2.5, Lemma 2.6, Remark 2.5, Remark 2.6 and Theorem 2.6 into account, yield the demonstration. □

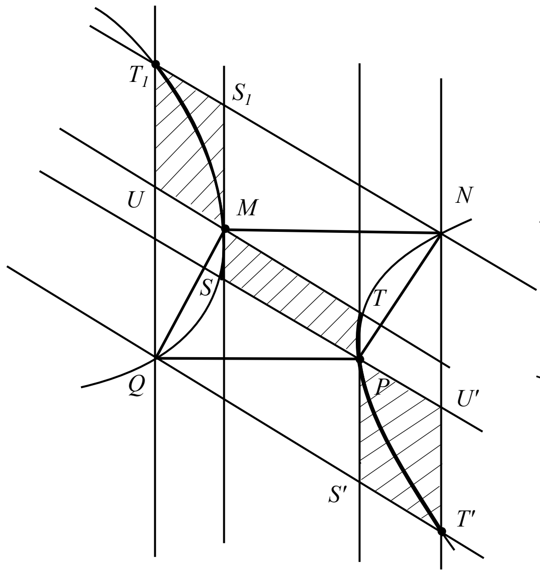

Fig. 2.18

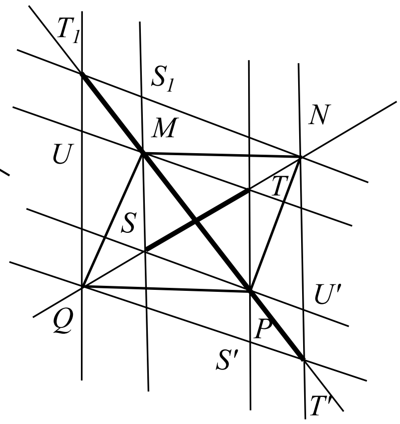

Fig. 2.19

Theorem 2.8. *Let $MNPQ$ be a given parallelogram, $M(c, b)$, $N(a, b)$, $P(-c, -b)$, $Q(-a, -b)$, $a > 0$, $b > 0$, $c < a$, $-c \leq a$, $c \geq 0$, $S_1V \perp MN$, $T_1U \perp MN$, $UV \perp MQ$, $T_1S_1 \perp MQ$, $N \in T_1S_1$, $M \in UV$, $Q \in T_1U$, $P \in VS_1$, $S'V' \perp QP$, $T'U' \perp QP$, $U'V' \perp PN$, $T'S' \perp PN$, $Q \in T'S'$, $P \in U'V'$, $N \in T'U'$, $M \in V'S'$* (Fig. 2.15)*, and ω a point situated in the exterior of the $MNPQ$ parallelogram, $\omega M \perp AB$, $\omega N \perp BC$, $\omega P \perp CD$ and $\omega Q \perp DA$.*
The $ABCD$ quadrilateral is cyclic and ω is the center of the circumscribed circle of the $ABCD$ quadrilateral if and only if

a) if $c \neq -\dfrac{b^2}{a}$, ω belongs to the intersection between the hyperbola (H) given by (2.26) *and the set $[S_1T_1UV] \cup [S'T'U'V']$* (Fig. 2.16)*;*

b) if $c = -\dfrac{b^2}{a}$ ω belongs to the set $[T_1V] \cup [V'T']$ (Fig. 2.16).

Proof. From Lemma 2.5, Theorem 2.5, Lemma 2.6, Remark 2.5, Remark 2.6 and Theorem 2.6 yield the proof. □

In the end, we withdraw the conclusion in Theorem 2.9.

Theorem 2.9. *Let $MNPQ$ be a give parallelogram, $M(c, b)$, $N(a, b)$, $P(-c, -b)$, $Q(-a, -b)$, $a > 0$, $b > 0$, $c < a$, $-c \leq a$, and ω a point situated in the plane, $\omega M \perp AB$, $\omega N \perp BC$, $\omega P \perp CD$ and $\omega Q \perp DA$.*

(i) Let $c < 0$ and $MS \perp PQ$, $PT \perp PQ$, $S, T \in \mathit{Int}\, MNPQ$, $MS_1 \perp MN$, $QT_1 \perp MN$, $PS' \perp PQ$, $NT' \perp PQ$, $MU \perp MQ$, $QS' \perp MQ$, $PU' \perp PN$, $NS_1 \perp PN$ (see Fig. 2.18).

Then the $ABCD$ quadrilateral is cyclic and ω is the center of the circumscribed circle of the $ABCD$ quadrilateral if and only if:

a) if $c \neq -\dfrac{b^2}{a}$, ω belongs to the set determinated by the intersection between the hyperbola (H) given by (2.4) and $[MSPT] \cup [MS_1T_1U] \cup [PS'T'U']$;

b) if $c = -\dfrac{b^2}{a}$ ω belongs to the set $[T_1T'] \cup [ST]$ (see Fig. 2.19*).*

(ii) Let $c \geq 0$ and $T_1U \perp PQ$, $S_1V \perp PQ$, $S'V' \perp MN$, $T'U' \perp MN$, $UV \perp PN$, $T_1S_1 \perp PN$, $S'T' \perp PN$, $U'V' \perp PN$ (see Fig. 2.20)*.*
Then the $ABCD$ quadrilateral is cyclic and ω is the center of the circumscribed circle of the $ABCD$ quadrilateral if and only if:

a) if $c \neq -\dfrac{b^2}{a}$, ω belongs to the intersection between hyperbola (H) given by (2.4) and the $[VS_1T_1U] \cup [V'S'T'U']$.

b) if $c = -\dfrac{b^2}{a}$, ω belongs to the set $[T_1V] \cup [S'T']$.

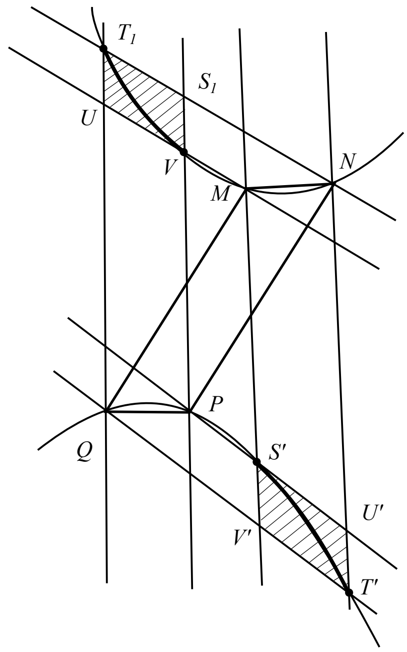

Fig. 2.20

In conclusion, for $c \neq -\dfrac{b^2}{a}$, $c < 0$, the hyperbola (H) given by (2.4) has got a branch that goes through the points T_1, M, S, Q and another branch goes through the points N, T, P, T'. For $c \geq 0$, a branch goes through the points T_1, V, M, N, and another branch goes through the points Q, P, S', T'.

Finally, we give the method of construction a figure determinated by a point of geometrical locus.

Let $MNPQ$ be a given Varignon's parallelogram and ω a point of the geometrical locus. The perpendicular in ω to ωM intersects the perpendicular in ω to ωQ in A, and similarly are obtained the points B, C, D (see Fig. 2.1). So, we get the cyclic quadrilateral $ABCD$, where ω is the center of the circumscribed circle.

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