INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 3 (2014), No. 1, 48 - 52

A COLLINEATION GROUP IN PROJECTIVE SPACE OF DIMENSION THREE AND DESMIC SYSTEM OF TETRAHEDRONS

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Abstract. In the present study given a pair of doubly perspective tetrahedrons (A) and (B), a collineation ω_1 is obtained under which the tetrahedrons (A) and (B) are fixed. When (A) and (B) becomes fourfold perspective (desmic), another collineation ω_2 is obtained under which the desmic pair $\{(A),(B)\}$ is fixed. The collineations ω_1 and, ω_2 generate a collineation group $\{e,\omega_1,\omega_2,\omega_3=\omega_1\omega_2\}$ under which the desmic system $\{(A),(B),(C)\}$ as well as its associated desmic system are fixed.

1. Notations and Preliminaries

Let P_3 be a three dimensional projective space over a formally real field $\mathfrak{F}$ [1]. A formally real field has characteristic equal to zero.
Let $(X)=X_1X_2X_3X_4, X=A,B$ be two tetrahedrons in P_3. The homogeneous coordinates of the vertices of (B) referred to (A) and some unit point forms the rows of a 4×4 matrix S [4]. The homogeneous coordinates (x_i) and (tx_i) for $i=1,2,3,4$ and $t(\neq 0)\in\mathfrak{F}$, represent the same point. So for every non singular diagonal matrix $D=\text{diag }(d_1,d_2,d_3,d_4), DS$ also represents the coordinates of the vertices of (B). The matrices S and T are defined to be equivalent if $T=DS$ for some nonsingular diagnoal matrix D. This equivalence relation gives equivalence classes, where rows of any matrix of a particular class represent the vertices of a particular tetrahedron for another. In matrix form it is written as $(B)_A=S=((b_{ij}))$ to represent (B) referred to (A). So if $(B)_A=((b_{ij})),(A)_C=((a_{ij}))$ then $(B)_C=((d_{ij}))=((b_{ij}))((a_{ij}))$ [3].

Keywords and phrases: Homogeneous coordinates, collineation, perspective tetrahedrons, desmic system of tetrahedrons.
(2010)Mathematics Subject Classification: 51N15
Received: 19.08.2013. In revised form: 11.12.2013. Accepted: 02.17.2014.

Note 1: The matrix of a tetrahedron can be multiplied by a non singular diagonal matrix from the left without changing the tetrahedron. But it cannot be done, when the matrix is used for the coordinate transformation as mentioned above. However the matrix of a tetrahedron can be multiplied by a nonzero scalar to make the matrix a simpler one, for a coordinate transformation [3].

Definition 1.1. *Two tetrahedrons (A) and (B) are called perspective if the lines A_iB_i (lines joining the vertices A_i and B_i) are concurrent at some point P. In this case the lines common to corresponding planes are on a plane $\pi.P$ is called as center of perspectivity and π is called as the plane of perspectivity.*

Here $(B)_A = \begin{bmatrix} a_1 & p_2 & p_3 & p_4 \\ p_1 & a_2 & p_3 & p_4 \\ p_1 & p_2 & a_3 & p_4 \\ p_1 & p_2 & p_3 & a_4 \end{bmatrix}$ which, can be denoted as $(B)_A = \dfrac{(a_1, a_2, a_3, a_4)}{p_1, p_2, p_3, p_4}$
where the parameters $a_1, a_2, a_3, a_4 \in \mathfrak{F}$ and the coordinates of P for (A) are (p_1, p_2, p_3, p_4)[3]. When the terahedrons (A) and (B) are desmic (four fold perspective) [2] then $(B)_A = \dfrac{(-p_1, p_2, p_3, p_4)}{(p_1, -p_2, -p_3, -p_4)}$ [3]. The four centers of perspectivities of these four perspectivities form the vertices of a third tetrahedron (C) given as $(C)_A = \dfrac{(-p_1, -p_2, -p_3, -p_4)}{(p_1, p_2, p_3, p_4)}$. $\{(A), (B), (C)\}$ forms a desmic system. When the tetrahedron (A) and (B) are doubly perspective we have $(B)_A = \dfrac{\left(\frac{P_1}{k}, \frac{P_2}{k}, p_3k, p_4k\right)}{(p_1, p_2, p_3, p_4)}$. Here for $k = -1(A)$ and (B) will be desmic.

Definition 1.2. *A transformation from a space P_3 to a space P_3' in which the relation of linear dependence is invariant is called a collineation* [5].

In other words under a collineation, collinear points are going to collinear points and, of course, coplanar points are going to coplanar points. In the present study we consider collineations where P_3' is same as P_3.

Definition 1.3. *If under a collineation α the vertices of a tetrahedron move into themselves then that tetrahedron is called a fixed tetrahedron under α.*

Definition 1.4. *If under a collineation α the vertices of a tetrahedron remains fixed then that tetrahedron is called a fvt (fixed vertex tetrahedron i.e. fixed vertex wise under α) of α.*

2. Collineation Group and Desmic System

Sanyal [4] has studied the following properties for the multi associated simplexes and perspective tetrahedrons.

(i) If (A) and (B) are two simplexes in projective space of dimension n and p is an $n+1$ cycle such that no vertex of (B) lies on any face of (A) and (A) is associated to both (B) and (Bp), then there exists a collineation of period $n+1$ under which both (A) and (B) are fixed.

(ii) If (A) and (B) are two perspective tetrahedrons in a projective space of dimension three and (A) is associated to (Bp) where p is any permutation over a set $\{1,2,3,4\}$ then (A) and (Bp^2) are perspective with the centers are images of each other under a cyclic collineation ω where (A) and (B) are invariant under ω.

In the present study similar results for multi perspective tetrahedrons are obtained in a P_3. It may be noted that multi perspectivity does not exist in projective space of dimension greater than 3.
Let $X = (x_1, x_2, x_3, x_4)$ be a point in p_3. Let ω be a transformation where $X\omega = X' = (x_1', x_2', x_3', x_4')$. If $x_i' = \lambda_j x_j, j = 1,2,3,4$ for $\lambda_j(\neq 0) \in \mathfrak{F}$, then it can be verified that ω is a collineation.
Let (A) and (B) be a pair of doubly perspective tetrahedrons as mentioned in definition 1.1, where (A) is the tetrahedron of reference. Let ω_1 be a collineation defined as

$$x_1' = \frac{p_1}{p_2}x_2, \quad x_2' = \frac{p_2}{p_1}x_1, \quad x_3' = \frac{p_3}{p_4}x_4, \quad x_4' = \frac{p_4}{p_3}x_3,$$

Here ω_1 is a collineation of period 2. It can be checked that

$$A_1\omega_1 = A_2, \qquad A_2\omega_1 = A_1, \qquad A_3\omega_1 = A_4, \qquad A_4\omega_1 = A_3$$

and

$$B_1\omega_1 = B_2, \qquad B_2\omega_1 = B_1, \qquad B_3\omega_1 = B_4, \qquad B_4\omega_1 = B_3.$$

So the two tetrahedrons (A) and (B) are fixed under the collineation ω_1. This collineation is of period 2 unlike the collineation obtained by Sanyal [4] which was a cyclic collineation of period n. This is because in the doubly perspective pairs (A) and (B), (A) is perspective with $(B) = B_1B_2B_3B_4$ and $(Bs) = B_2B_1B_4B_3$, where s is a permutation of order 2. So we have obtained the following lemma.

Lemma 2.1. *Given a pair of doubly perspective tetrahedrons (A) and (B), there exists a collineation ω_1 of period 2 depending on the permutation s for which (A) is perspective with (B) and (Bs).*

Let (A) is also perspective with $B_3B_4B_1B_2$. In that case the value of the parameter k will be -1 and (A) and (B) will be in fourfold perspective i.e. form a desmic pair and is given by $(B)_A = \dfrac{(-p_1, -p_2, -p_3, -p_4)}{(p_1, p_2, p_3, p_4)}$. Let us define a collineation ω_2 as follows

$$x_1' = \frac{p_1}{p_3}x_3, \quad x_2' = \frac{p_2}{p_4}x_4, \quad x_3' = \frac{p_3}{p_1}x_1, \quad x_4' = \frac{p_4}{p_2}x_2,$$

It can be checked that

$$A_1\omega_2 = A_3, \qquad A_2\omega_2 = A_4, \qquad A_3\omega_2 = A_1, \qquad A_4\omega_2 = A_2$$

and

$$B_1\omega_2 = B_3, \qquad B_2\omega_2 = B_4, \qquad B_3\omega_2 = B_1, \qquad B_4\omega_2 = B_2$$

Here ω_2 is also a collineation of period 2 under which the tetrahedrons (A) and (B) are fixed. The collineation ω_1 along with ω_2 generates a group of

order given as $\{e, \omega_1, \omega_2, \omega_3 = \omega_1\omega_2\}$
Where $\omega_1\omega_2$, denoted as ω_3, is a collineation expressed as follows

$$x_1' = \frac{p_1}{p_4}x_4, \quad x_2' = \frac{p_2}{p_3}x_3, \quad x_3' = \frac{p_3}{p_2}x_2, \quad x_4' = \frac{p_4}{p_1}x_1,$$

It can be checked that

$$A_1\omega_3 = A_4, \qquad A_2\omega_3 = A_3, \qquad A_3\omega_3 = A_2, \qquad A_4\omega_3 = A_1$$

and

$$B_1\omega_3 = B_4, \qquad B_2\omega_3 = B_3, \qquad B_3\omega_3 = B_2, \qquad B_4\omega_3 = B_1$$

Let (C) be the third tetrahedron of the desmic system where $(C)_A = \dfrac{(-p_1, p_2, p_3, p_4)}{(p_1, -p_2, -p_3, -p_4)}$. It can be verified that

$$C_1\omega_1 = C_1, C_2\omega_1 = C_2, C_3\omega_1 = C_3, C_4\omega_1 = C_4$$

$$C_1\omega_2 = C_1, C_2\omega_2 = C_2, C_3\omega_2 = C_3, C_4\omega_2 = C_4$$

$$C_1\omega_3 = C_1, C_2\omega_3 = C_2, C_3\omega_3 = C_3, C_4\omega_3 = C_4.$$

So the tetrahedrons $(A), (B)$ and (C) are fixed under each of the collineation in the group so obtained. This gives us the following theorem.

Theorem 2.1. *If $(A), (B)$ and (C) are the tetrahedrons in a desmic system, then there exists a group of collineations with 4 elements $\{e, \omega_1, \omega_2, \omega_3 = \omega_1\omega_2\}$ so that each tetrahedron of this system is fixed under the collineation group.*

Associated Desmic System: Given a desmic system $(A) = A_1A_2A_3A_4, (B) = B_1B_2B_3B_4$ and $(C) = C_1C_2C_3C_4$. The three pairs of opposite edges of one tetrahedron intersect the three pair of opposite edges of another tetrahedron in 12 distinct points. These set of 12 points is same for any desmic pair of the desmic system $\{(A), (B), (C)\}$. These 12 points when grouped suitably form 3 tetrahedrons of another desmic system $\{(P), (Q), (R)\}$ called as the associated desmic system to $\{(A), (B), (C)\}$[2].
For the desmic system $\{(A), (B), (C)\}$ as considered here the associated desmic system is obtained as

$$(P)_A = \begin{bmatrix} -p_1 & 0 & 0 & p_4 \\ 0 & p_2 & p_3 & 0 \\ p_1 & 0 & 0 & p_4 \\ 0 & -p_2 & p_3 & 0 \end{bmatrix}, (Q)_A = \begin{bmatrix} 0 & -p_2 & 0 & p_4 \\ p_1 & 0 & p_3 & 0 \\ 0 & p_2 & 0 & p_4 \\ p_1 & 0 & -p_3 & 0 \end{bmatrix}.$$

And $(R)_A = \begin{bmatrix} 0 & 0 & -p_3 & p_4 \\ p_1 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & p_4 \\ -p_1 & p_2 & 0 & 0 \end{bmatrix}$

By changing the reference tetrahedron from (A) to (P), the tetrahedrons (Q) and (R) can be expressed as follows.
$(Q)_p = \dfrac{(1, 1, -1, -1)}{(-1, -1, 1, 1)}$ and $(R)_p = \dfrac{(1, 1, -1, 1)}{(-1, -1, 1, -1)}$
After comparing these expressions with the expressions of $(B)_A$ and $(C)_A$

in definition 1.1 justifies that $\{(P),(Q),(R)\}$ forms a desmic system.
It can be verified that

$$\begin{array}{llll} P_1\omega_1 = P_4 & P_2\omega_1 = P_3 & P_3\omega_1 = P_2 & P_4\omega_1 = P_1, \\ P_1\omega_2 = P_4 & P_2\omega_2 = P_3 & P_3\omega_2 = P_2 & P_4\omega_2 = P_1, \\ P_1\omega_3 = P_1 & P_2\omega_3 = P_2 & P_3\omega_3 = P_3 & P_4\omega_3 = P_4. \end{array}$$

It can be observed that though the vertices of the tetrahedron (P) are not fixed under the collineations ω_1 and ω_2 the tetrahedron itself fixed under them. In the contrary the vertices of (P) are fixed under the collineation ω_3. So (P) is a fvt of ω_3.Similarly it can be observed that

$$\begin{array}{llll} Q_1\omega_1 = Q_4 & Q_2\omega_1 = Q_3 & Q_3\omega_1 = Q_2 & Q_4\omega_1 = Q_1, \\ Q_1\omega_2 = Q_1 & Q_2\omega_2 = Q_2 & Q_3\omega_2 = Q_3 & Q_4\omega_2 = Q_4, \\ Q_1\omega_3 = Q_4 & Q_2\omega_3 = Q_3 & Q_3\omega_3 = Q_2 & Q_4\omega_3 = Q_1, \\ R_1\omega_1 = R_1 & R_2\omega_1 = R_2 & R_3\omega_1 = R_3 & R_4\omega_1 = R_4, \\ R_1\omega_2 = R_4 & R_2\omega_2 = R_3 & R_3\omega_2 = R_2 & R_4\omega_2 = R_1, \\ R_1\omega_3 = R_4 & R_2\omega_3 = R_3 & R_3\omega_3 = R_2 & R_4\omega_3 = R_1. \end{array}$$

The above observation shows that the tetrahedron (Q) is fixed under the collineations $\omega_1$and ω_3 and it is a fvt of ω_2. Similarly we have the tetrahedron (R) fixed under ω_2 and ω_3 while it is a fvt of ω_1. So we arrive at the following theorem.

Theorem 2.2. *The associated desmic system $\{(P),(Q),(R)\}$ to the desmic system $\{(A),(B),(C)\}$ as mentioned in theorem 2.2 is fixed under the collineation group $\{e,\omega_1,\omega_2,\omega_3 = \omega_1\omega_2\}$ where $(P),(Q)$ and (R) are fvt under ω_3,ω_2 and ω_1 respectively .*

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