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INEQUALITIES FOR A SIMPLEX

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Abstract. In this note, a generalization of G. Apostolopoulos' problem in the n-dimensional space are given.

1. Introduction

In 2014 G. Apostolopoulos [1] proposed following problem:

Let ABC be a triangle with incentre I through which an arbitrary line passes meeting sides AB and AC at the point D and E respectively. Show that

$$\frac{1}{r} \geq \frac{1}{AD} + \frac{1}{AE}$$

where r denotes the inradius of ABC.

The main aim of this note is to show the following theorem. As a consequence of the Theorem 2.1, G.Apostolopoulos' problem is proved.

2. Main result

Lemma 2.1. *Let us consider a simplex $B_1B_2\dots B_{n+1}$ in the n dimensional Euclidian space E^n. If the point M belongs to the simplex $B_1B_2\dots B_{n+1}$, then there exists a real numbers $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ such that $\overrightarrow{XM} = \sum_{k=1}^{n+1} \alpha_k \overrightarrow{XB}_k$ and $\alpha_1 + \cdots + \alpha_{n+1} = 1$ for arbitrary point X.*

Proof. Since the vectors $\overrightarrow{MB}_1, \overrightarrow{MB}_2, \dots, \overrightarrow{MB}_{n+1}$ linearly dependent there exists at least one is nonzero a real numbers $\beta_1, \beta_2, \dots, \beta_{n+1}$ such that

$$\beta_1\overrightarrow{MB}_1 + \beta_2\overrightarrow{MB}_2 + \cdots + \beta_{n+1}\overrightarrow{MB}_{n+1} = 0. \tag{1}$$

Since $\overrightarrow{MB}_k = \overrightarrow{XB}_k - \overrightarrow{XM}$, from (1) we have

$$(\beta_1 + \beta_2 + \cdots + \beta_{n+1}) \cdot \overrightarrow{XM} = \beta_1\overrightarrow{XB}_1 + \beta_2\overrightarrow{XB}_2 + \cdots + \beta_{n+1}\overrightarrow{XB}_{n+1}.$$

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Finally we choose the desired real numbers such that

$$\alpha_k = \frac{\beta_k}{\beta_1 + \beta_2 + \cdots + \beta_{n+1}}$$

for $k \in \{1, 2, \ldots, n+1\}$. □

Lemma 2.2. *Let us consider a simplex $A_1A_2 \ldots A_{n+1}$ and the hyperplane $A_1A_2 \ldots A_n$ in the n dimensional Euclidian space E^n. If we denote by V_n, V_{n-1} the volumes of the simplex and the hyperplane, respectively, then $V_n = \frac{1}{n}V_{n-1}h$, where h is length of height from the vertex A_{k+1} to the hyperplane $A_1A_2 \cdots A_n$*

Proof. We have:

$$V_n = \int_0^h \left(\frac{t}{h}\right)^{n-1} V_{n-1}dt = \frac{V_{n-1}}{h^{n-1}} \int_0^h t^{n-1}dt = \frac{V_{n-1}}{h^{n-1}} \cdot \frac{h^n}{n} = \frac{1}{n}V_{n-1}h.$$

□

Theorem 2.1. *Let $A = A_1A_2 \cdots A_nA_{n+1}$ be a simplex in the n dimensional Euclidian space E^n, I be the center of inscribed sphere in this simplex. Let $B_1, B_2, \cdots, B_n$ be the points for which a hyperplane crossing the point I intersecting with $A_{n+1}A_1, A_{n+1}A_2, \cdots, A_{n+1}A_n$, respectively. Then*

$$\frac{1}{A_{n+1}B_1} + \frac{1}{A_{n+1}B_2} + \cdots + \frac{1}{A_{n+1}B_n} \leq \frac{1}{r},$$

where r is the radius of inscribed sphere of $A = A_1A_2 \cdots A_{n+1}$.

Proof. Let $V_{A_1 \cdots A_{i-1}A_{i+1} \cdots A_{n+1}} = v_i,\ i = 1, 2, \cdots, n+1$. We know that I is the center of mass the points $v_1A_1, v_2A_2, \cdots, v_{n+1}A_{n+1}$. Hence

$$\forall X \in E^n : \overrightarrow{XI} = \sum_{k=1}^{n+1} \frac{v_k}{\sum_{i=1}^{n+1} v_i} \cdot \overrightarrow{XA_k}$$

$$= \frac{v_{n+1}}{\sum_{i=1}^{n+1} v_i} \cdot \overrightarrow{XA_{n+1}} + \frac{\sum_{i=1}^{n} v_i}{\sum_{i=1}^{n+1} v_i} \cdot \sum_{k=1}^{n} \frac{v_k}{\sum_{i=1}^{n} v_i} \cdot \overrightarrow{XA_k} \tag{2}$$

Let us denote by Q intersection of $A_{n+1}I$ with the $n-1$ dimensional hyperplane $A_1A_2 \cdots A_n$. Then Q will be the center of mass the points $v_1A_1, v_2A_2, \cdots, v_nA_n$. Therefore

$$\forall X \in E^n : \overrightarrow{XQ} = \sum_{k=1}^{n} \frac{v_k}{\sum_{i=1}^{n} v_i} \cdot \overrightarrow{XA}_k \tag{3}$$

From (2) and (3) we have,

$$\forall X \in E^n : \overrightarrow{XI} = \frac{v_{n+1}}{\sum_{i=1}^{n+1} v_i} \cdot \overrightarrow{XA}_{n+1} + \frac{\sum_{i=1}^{n} v_i}{\sum_{i=1}^{n+1} v_i} \cdot \overrightarrow{XQ} \tag{4}$$

Let

$$\overrightarrow{A_{n+1}A_k} = \overrightarrow{e}_k,\ k = 1, 2, \cdots, n \text{ and } \frac{A_{n+1}A_k}{A_{n+1}B_k} = \frac{1}{b_k},\ k = 1, 2, \cdots, n,\ \frac{A_{n+1}Q}{A_{n+1}I} = \frac{1}{b}.$$

From (3) we have, $\overrightarrow{A_{n+1}Q} = \sum_{k=1}^{n} \frac{v_k}{\sum_{i=1}^{n} v_i} \cdot \overrightarrow{A_{n+1}A_k}$. Hence

$$\overrightarrow{A_{n+1}I} = b \cdot \overrightarrow{A_{n+1}Q} = \sum_{k=1}^{n} \frac{bv_k}{\sum_{i=1}^{n} v_i} \cdot \overrightarrow{A_{n+1}A_k}$$

(5)
$$= \sum_{k=1}^{n} \frac{bv_k}{\sum_{i=1}^{n} v_i} \cdot \overrightarrow{e}_k$$

Since I belongs to the simplex $B_1B_2\cdots B_n$, by the Lemma 2.1 there exists real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that

(6)
$$\overrightarrow{A_{n+1}I} = \sum_{k=1}^{n} \alpha_k \overrightarrow{A_{n+1}B_k} = \sum_{k=1}^{n} \alpha_k b_k \overrightarrow{e}_k \text{ and } \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

Since the basis is unique, also by (5) and (6) we have,

$$\begin{cases} \frac{bv_1}{\sum_{i=1}^{n} v_i} & = \alpha_1 b_1 \\ \frac{bv_2}{\sum_{i=1}^{n} v_i} & = \alpha_2 b_2 \\ & \vdots \\ \frac{bv_n}{\sum_{i=1}^{n} v_i} & = \alpha_n b_n \end{cases} \Leftrightarrow \begin{cases} \frac{bv_1}{b_1 \cdot \sum_{i=1}^{n} v_i} & = \alpha_1 \\ \frac{bv_2}{b_2 \cdot \sum_{i=1}^{n} v_i} & = \alpha_2 \\ & \vdots \\ \frac{bv_n}{b_n \cdot \sum_{i=1}^{n} v_i} & = \alpha_n \end{cases}$$

Consequently,

$$\frac{bv_1}{b_1 \cdot \sum_{i=1}^{n} v_i} + \frac{bv_2}{b_2 \cdot \sum_{i=1}^{n} v_i} + \cdots + \frac{bv_n}{b_n \cdot \sum_{i=1}^{n} v_i} = 1$$

$$\Leftrightarrow \frac{1}{b} = \frac{v_1}{\sum_{i=1}^{n} v_i} \cdot \frac{1}{b_1} + \frac{v_2}{\sum_{i=1}^{n} v_i} \cdot \frac{1}{b_2} + \cdots + \frac{v_n}{\sum_{i=1}^{n} v_i} \cdot \frac{1}{b_n}$$

(7)
$$\Leftrightarrow \frac{A_{n+1}Q}{A_{n+1}I} = \frac{v_1}{\sum_{i=1}^{n} v_i} \cdot \frac{A_{n+1}A_1}{A_{n+1}B_1} + \frac{v_2}{\sum_{i=1}^{n} v_i} \cdot \frac{A_{n+1}A_2}{A_{n+1}B_2} + \cdots + \frac{v_n}{\sum_{i=1}^{n} v_i} \cdot \frac{A_{n+1}A_n}{A_{n+1}B_n}$$

From the other hand, from (4) we get,

(8)
$$\frac{A_{n+1}Q}{A_{n+1}I} = \frac{\sum_{i=1}^{n+1} v_i}{\sum_{i=1}^{n} v_i}$$

Using (7) and (8) we have,

$$\frac{\sum_{i=1}^{n+1} v_i}{\sum_{i=1}^{n} v_i} = \frac{1}{\sum_{i=1}^{n} v_i} \cdot \sum_{k=1}^{n} v_k \cdot \frac{A_{n+1}A_k}{A_{n+1}B_k}$$

(9)
$$\Leftrightarrow \sum_{i=1}^{n+1} v_i = \sum_{k=1}^{n} v_k \cdot \frac{A_{n+1}A_k}{A_{n+1}B_k}$$

If h_k is length of height from the vertex A_k to the polytope $A_1 \cdots A_{k-1}A_{k+1} \cdots A_{n+1}$, then $A_{n+1}A_k \geq h_k,\ k = 1, 2, \cdots, n$. Applying the last inequality to (9) and by the Lemma 2.2 we have

$$\sum_{i=1}^{n+1} v_i \geq \sum_{k=1}^{n} \frac{v_k h_k}{A_{n+1}B_k} = \sum_{k=1}^{n} \frac{nV}{A_{n+1}B_k}$$
$$\Leftrightarrow \sum_{k=1}^{n} \frac{1}{A_{n+1}B_k} \leq \frac{\sum_{i=1}^{n} v_i}{nV} = \frac{1}{r},$$

where V is the volume of the simplex $A_1A_2 \cdots A_nA_{n+1}$. The proof is completed. □

References

[1] Apostolopoulos, G., Problem 3807, Crux Mathematicorum, **39(1)(2013)**.

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