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ELLIPTIC INVERSION OF TWO-DIMENSIONAL OBJECTS [A graphical point of view with *Mathematica*]

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Abstract. In this paper we explore the basic properties of the inversion in an ellipse, which generalizes the classical inversion with respect to a circle, from a computational point of view. Topics included in this paper are the elliptic inversion of lines, circles, ellipses and some parametric curves. We use the software *Mathematica* for computing and displaying those two-dimensional objects. The output obtained is consistent with Mathematica's notation and results.

1. Introduction

Given any point O in the plane and any real positive number r (the constant of inversion), the *inversion image* of a point P is the point P', taken from the ray $\overrightarrow{OP}$, and such that the product of the distances OP and OP' satisfies $OP \cdot OP'^2$. In this case, the circle of radius r is invariant under this geometrical operation and is the border of a region in the plane for the which, points inside the region are moved to points outside and vice versa.

The circle inversion was invented by J. Steiner about 1830 [11]. It has been widely described in the literature (see, e.g., [2, 3, 5, 6, 7, 10, 12]) and has inspired beautiful books such as Indra's Pearls [8], Visual Complex Analysis [9] and movies as the film *Möbius Transformations Revealed* [1]. Moreover, authors in [13] and [15] have developed *Mathematica* packages for computing and displaying images by the circle inversion transformation.

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On the other hand, circle inversion transformation can be generalized by taking other objects, such as parallel lines [14] or central conics [4, 16]. In particular, if the plane is divided in two regions by an ellipse, the constant of inversion is not constant, since it will change according to the direction of the ray $\overrightarrow{OP}$.

In this paper, we study and show some codes in the software Mathematica for computing and displaying the images of two-dimensional objects by the inversion in an ellipse.

2. Elliptic Inversion

Definition 2.1. *Let $\mathcal{E}$ be an ellipse centered at a point O in $\mathbb{R}^2$, the elliptic inversion in this ellipse is the mapping $\psi : \mathbb{R}^2 \setminus \{O\} \longmapsto \mathbb{R}^2 \setminus \{O\}$ defined by $\psi(P) = P'$, where P' lies on the ray $\overrightarrow{OP}$ and $OP \cdot OP'^2$, where Q is the point of intersection of the ray $\overrightarrow{OP}$ and the ellipse $\mathcal{E}$.*

The point P' is said to be the *elliptic inverse* of P in the ellipse $\mathcal{E}$ or with respect to the ellipse $\mathcal{E}$; $\mathcal{E}$ is called the *ellipse of inversion*, O is called the *center of inversion*, and the number OQ is called the *radius of inversion*, see Figure 1. Unlike the classical case, here the radius is not constant.

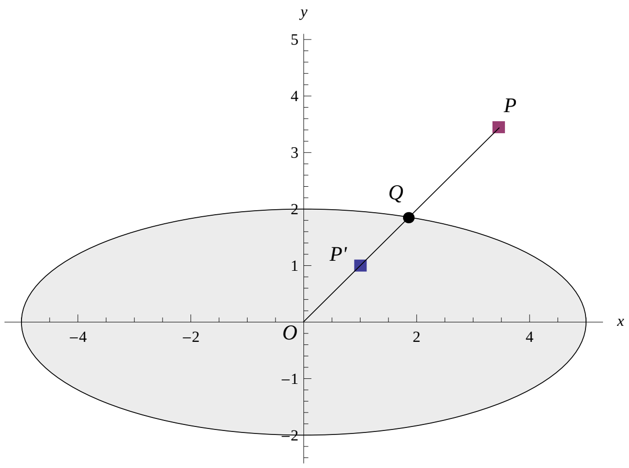


Figure 1. Inversion in an ellipse

The elliptic inversion is an involutive mapping, i.e., $\psi(\psi(P)) = P$. The fixed points are the points on the ellipse $\mathcal{E}$. Indeed, if F is a fixed point, $\psi(F) = F$, then $OF \cdot OF = (OF)^2 = (OQ)^2$. Hence $OF = OQ$ and as Q lies on the ray $\overrightarrow{OF}$, then $F = Q$. Moreover, it is clear that if P is in the exterior of $\mathcal{E}$ then P' is interior to $\mathcal{E}$, and conversely.

3. Inversion in an Ellipse with Mathematica

For simplicity suppose the ellipse has the equation $\mathcal{E}_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The proof of following theorems and corollaries can be found in [4, 16].

Theorem 3.1. *Let $P = (u, v)$ and $P' = (x, y)$ be a pair of elliptic points with respect to $\mathcal{E}_{a,b}$. Then*

$$x = \frac{a^2b^2u}{b^2u^2 + a^2v^2} \qquad y = \frac{a^2b^2v}{b^2u^2 + a^2v^2}$$

If $a = b = 1$, i.e., when $\mathcal{E}_{a,b}$ is a circle, we obtain the classical inversion transformation

$$\psi : (u, v) \longmapsto \left(\frac{u}{v^2 + u^2}, \frac{v}{v^2 + u^2}\right)$$

which coincides with the Möebius transformation $f(z) = \frac{1}{z}$.

The explicit cartesian coordinates for inverting points in an ellipse $\mathcal{E}_{a,b}$ is implemented by the following code `inversionElliptic`.

```
In[1]:=inversionElliptic[p : {_, _}, a_, b_] :=
Module[{x = p[[1]],y = p[[2]]}, {(a^2*b^2*x)/(b^2*x^2 + a^2*y^2),
(a^2*b^2*y)/(b^2*x^2 + a^2*y^2)}]
```

Example 3.1. *The next* `Manipulate` *code shows the inversion in an ellipse $\mathcal{E}_{a,b}$. Drag a point; the arrow starts at point and ends at ψ of this point. Moreover, you can change the values a and b, see Figure 2.*

```
In[2]:= Manipulate[Module[{x, y, Ez},{x, y} = z;
Ez = inversionElliptic[z, a, b];
Graphics[{Circle[{0, 0}, {a, b}], Arrow[{z, Ez}],
{Red, Disk[Ez, .1]}}, Axes -> True, PlotRange -> 5,
Prolog -> {Opacity[0.15], Gray, Disk[{0, 0}, {a, b}]}]],
{{z, {4, 3}}, Locator},
{{a, 3}, 1, 10, 1,Appearance -> "Labeled"},
{{b, 1}, 1, 10, 1, Appearance -> "Labeled"}]
Out[2]:=  See Figure 2
```

4. Elliptic Inversion of Some Curves

In this section, we explore the inversion in an ellipse of lines, ellipses and some parametric curves. If a point P moves on a curve $\mathcal{C}$, and P', the elliptic inverse of P with respect to $\mathcal{E}$ moves on a curve $\mathcal{C}'$, the curve $\mathcal{C}' = \psi(\mathcal{C})$ is called the *elliptic inverse* of $\mathcal{C}$. It is evident that $\mathcal{C}$ is the elliptic inverse of $\mathcal{C}'$ respect to $\mathcal{E}$.

Definition 4.1. *If two ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$ have parallel axes and have equal eccentricities, then they are said to be of the same semi-form. If in addition the principal axes are parallel, then they are called homothetic and it is denoted by $\mathcal{E}_1 \sim \mathcal{E}_2$.*

The proof of following theorems and corollaries can be found in [4, 16].

Theorem 4.1.
1. *The elliptic inverse of a line l which pass through the center of inversion is the line itself.*
2. *The elliptic inverse of a line l which does not pass through the center of inversion is a homothetic ellipse, which pass through the center of inversion and it is homothetic to the ellipse of inversion.*

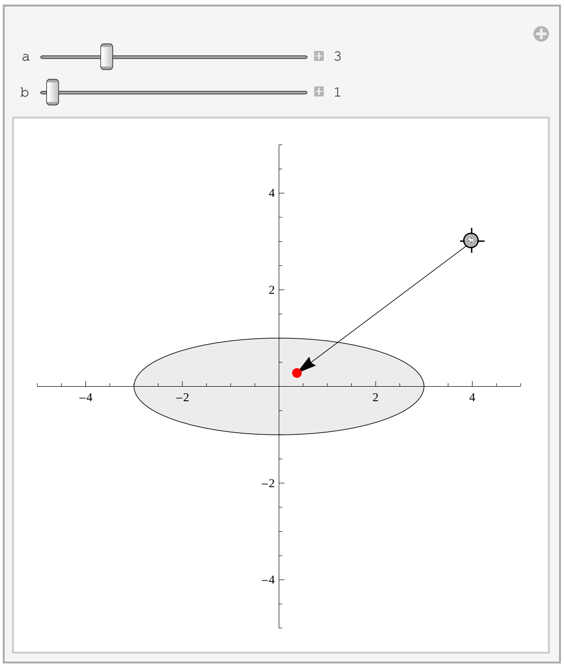


FIGURE 2. Inversion of a point in an ellipse

The next function **inverseELine** draw the inverse in an ellipse of a line $y = mx + d$ respect to $\mathcal{E}_{a,b}$.

```
In[3]:=inverseELine[m_, d_, a_, b_] :=ParametricPlot[{{t, m*t + d},
inversionElliptic[{t, m*t + d}, a, b]} // Evaluate,
{t, -100, 100}, AxesLabel -> {x, y},
PlotStyle -> {{Thickness[0.006], Blue},
{Thickness[0.006], Red}}, PlotRange -> {{-10, 10}, {-10, 10}},
PlotStyle -> AbsoluteThickness[9],
Prolog -> {Opacity[0.15], Gray, Disk[{0, 0}, {a, b}]},
Epilog -> {Black, Circle[{0, 0}, {a, b}]}]
In[4]:=inserseEline[1,3,5,3]
Out[4]:=See Figure 3.
```

Corollary 4.1. *Let l_1 and l_2 be perpendicular lines. Then*

(1) *If the intersecting point is different of the center of inversion, then $\psi(l_1)$ and $\psi(l_2)$ are orthogonal ellipses (their tangents at the points of intersection are perpendicular), see Figure 4 left.*
(2) *If the intersecting point is equal to the center of inversion, then $\psi(l_1)$ and $\psi(l_2)$ are perpendicular lines.*
(3) *If l_1 through the center of inversion but l_2 not through the center of inversion, then $\psi(l_1)$ is an ellipse and $\psi(l_2)$ is an line which passes through the center of inversion and it is orthogonal to $\psi(l_1)$ in the center of inversion, see Figure 4 right.*

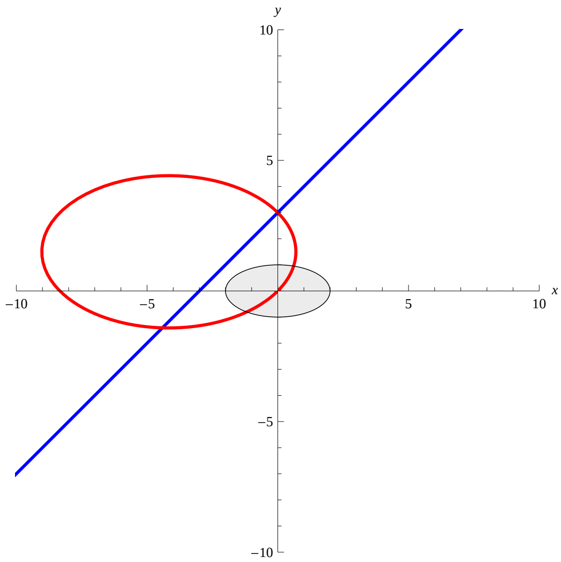


Figure 3. Elliptic inverse of a line l which does not pass through the center of the elliptic inversion.

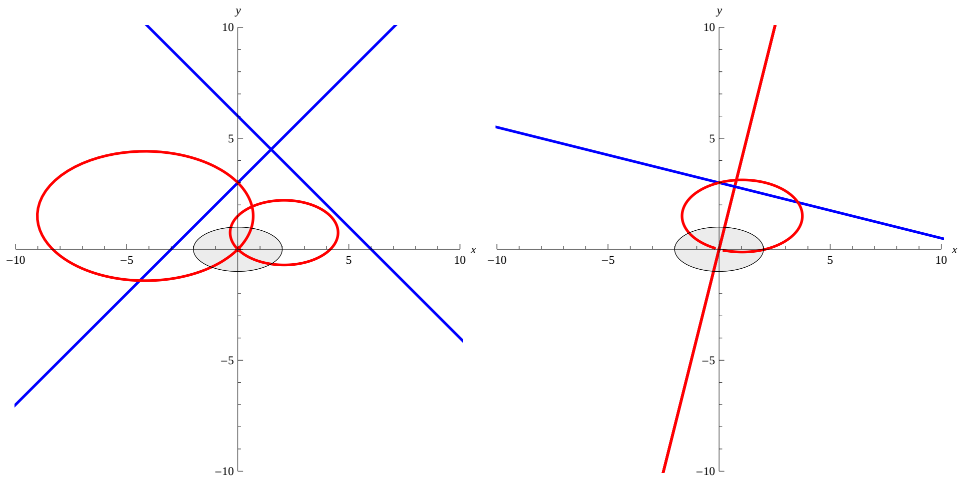


Figure 4. Elliptic inverse of perpendicular lines, Corollary 4.1.

Corollary 4.2. *The inversion in an ellipse of a system of concurrent lines for a point H, distinct of the center of inversion is a coaxial system of ellipses with two common points H' and the center of inversion, see Figure 5.*

Corollary 4.3. *The inversion in an ellipse of a system of parallel lines which does not pass through of the center of inversion is a set of tangent ellipses at the center of inversion, see Figure 6.*

Example 4.1. *In Figure 7, we show a grid and its inverse elliptic curve respecto to $\mathcal{E}_{2,1}$. This is the code:*

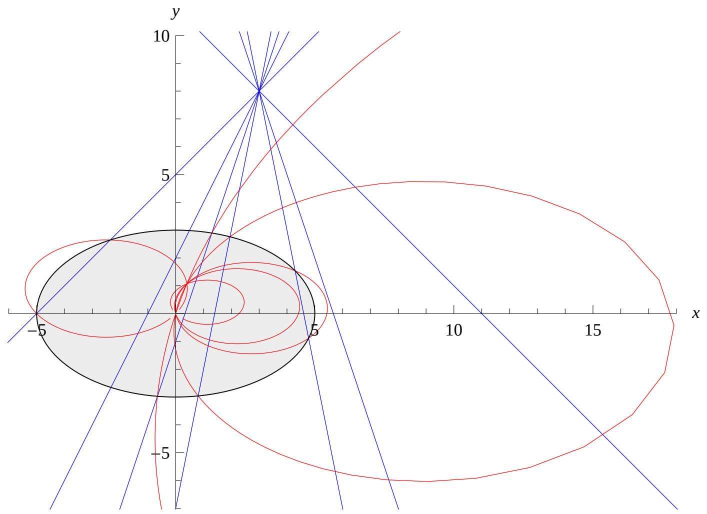


Figure 5. Inversion in an ellipse of a system of concurrent lines

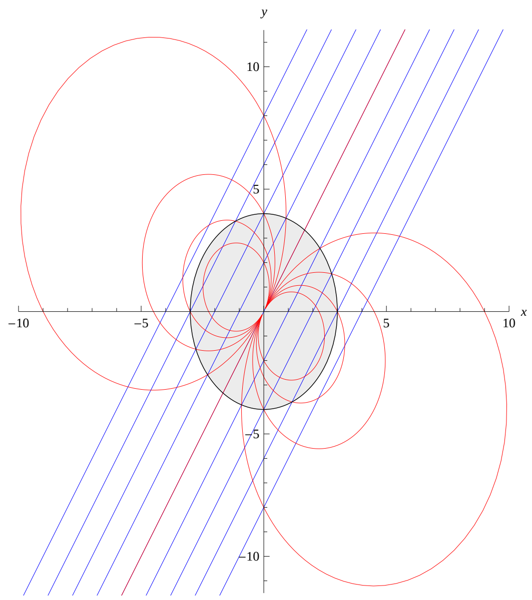


Figure 6. Inversion in an ellipse of a system of parallel lines

```
In[5]:=A = Table[inversionElliptic[{t, k}, 2, 1], {k, -4, 4, .5}];
B = Join[Table[inversionElliptic[{k, t}, 2, 1], {k, -4, 4, .5}], A];
ParametricPlot[B // Evaluate, {t, -35, 35},
PlotRange -> {{-2.7, 2.7}, {-2.7, 2.7}},
PlotStyle -> Thickness[0.001], ImageSize -> 500,
Epilog -> Circle[{0, 0}, {2, 1}]]
Out[5]:See Figure 5
```

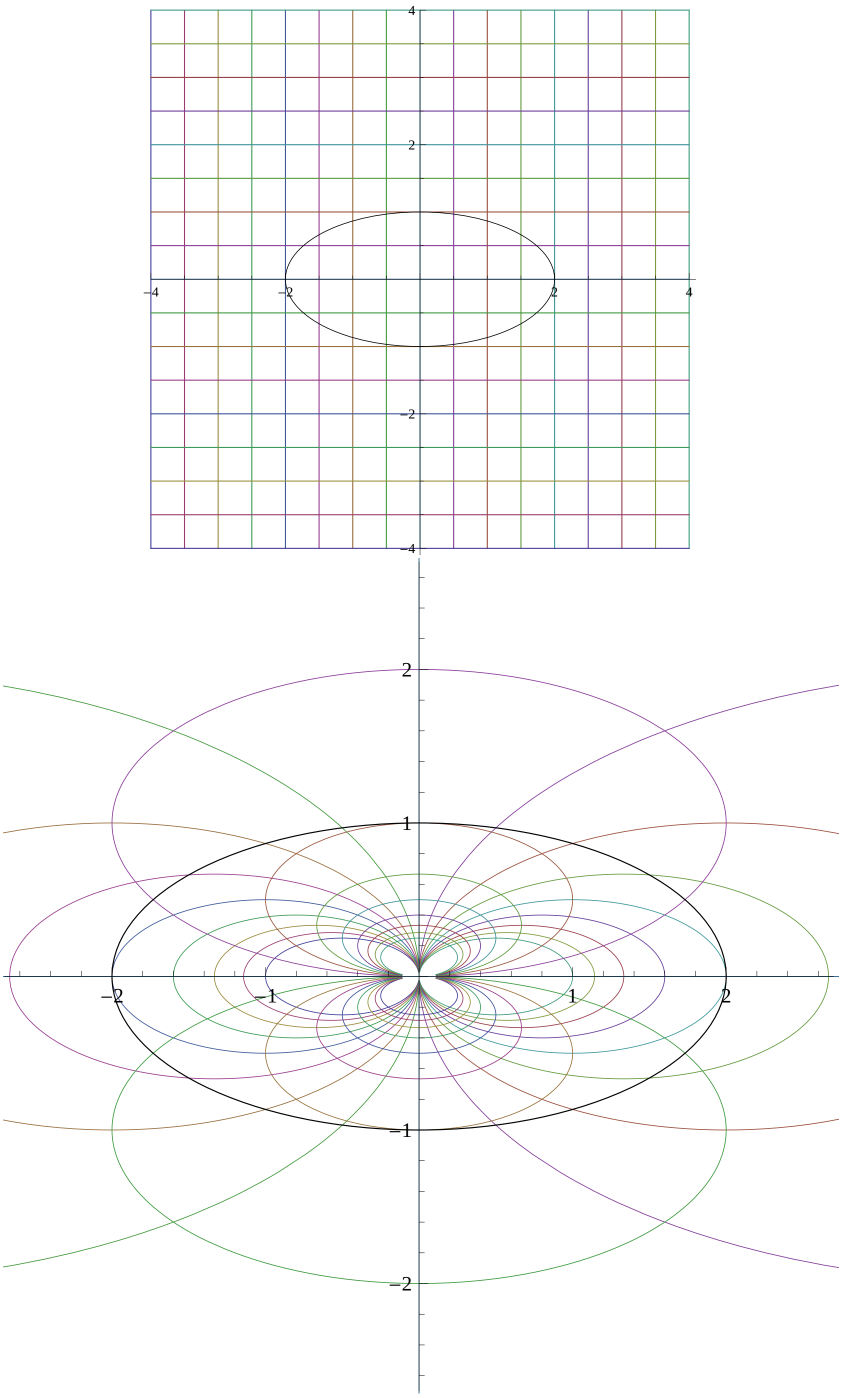


FIGURE 7. Inversion in an Ellipse of a Grid

Theorem 4.2. *Let χ and χ' be an ellipse and its elliptic inverse curve with respect to $\mathcal{E}_{a,b}$ such that $\chi \sim \mathcal{E}_{a,b}$.*

(1) If χ not passing through the center of inversion, then χ' is an ellipse not passing through the center of inversion and, $\chi' \sim \mathcal{E}_{a,b}$, see Figure 8 left.

(2) If χ passing through the center of inversion, then χ' is a line, see Figure 8 right.

(3) If χ is orthogonal to $\mathcal{E}_{a,b}$, then χ' is the ellipse itself.

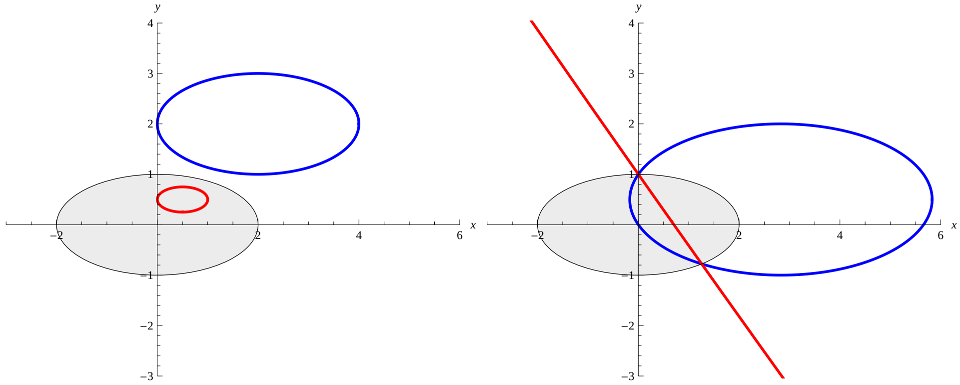


FIGURE 8. Theorem 4.2, Case *i* and *ii*.

Theorem 4.3. *The inverse of any conic not of the same semi-form as the ellipse of inversion and passing through the center of inversion is a cubic curve. The inverse of any conic not of the same semi-form as the ellipse of inversion and not passing through the center of inversion is a curve of the fourth degree.*

Example 4.2. *Consider the circumference of parametric equation*

$$\begin{cases} x &= 1.5\cos(t), \\ y &= 2 + 1.5\sin(t) \end{cases}$$

with $t \in [0, 2\pi]$, we obtain the inversion respect to $\mathcal{E}_{2,1}$, with the following code:

```
In[6]:=A = {1.5 Cos[t], (2 + 1.5 Sin[t])};
B = inversionElliptic[A, 2, 1];
ParametricPlot[{A, B} // Evaluate, {t, 0, 2 Pi}, AxesLabel -> {x, y},
PlotStyle -> {{Thickness[0.006], Blue}, {Thickness[0.006], Red}},
PlotRange -> {{-3, 3}, {-1.5, 4}},
PlotStyle -> AbsoluteThickness[9],
Prolog -> {Opacity[0.15], Gray, Disk[{0, 0}, {2, 1}]},
Epilog -> {Black, Circle[{0, 0}, {2, 1}]}]
Out[6]:= See Figure 9
```

Moreover, its parametric equations can also be obtained:

```
In[7]:=B
```

Out[7]= $\left\{ \dfrac{6.\cos(t)}{4(1.5\sin(t)+2)^2 + 2.25\cos^2(t)}, \dfrac{4(1.5\sin(t)+2)}{4(1.5\sin(t)+2)^2 + 2.25\cos^2(t)} \right\}$

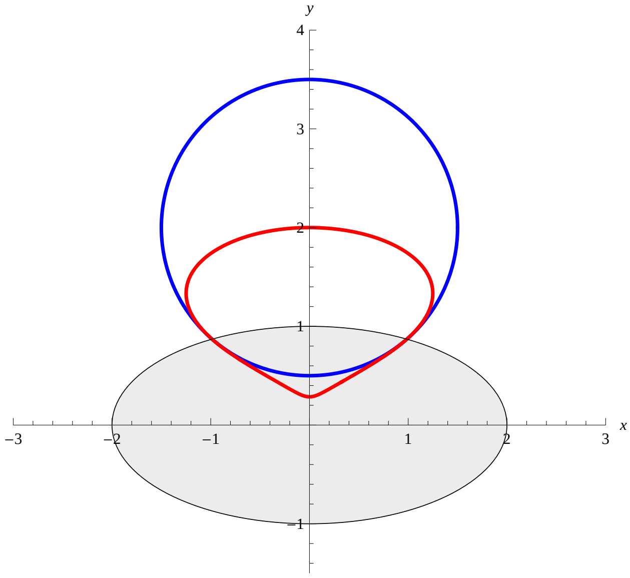


Figure 9. Elliptic Inversion of a Circumference.

Example 4.3. *The following* `Manipulate` *shows the elliptic inversion respect to* $\mathcal{E}_{a,b}$ *of any ellipse, see Figure 10, with parametric equations,*

$$\begin{cases} X_0 + a_0\cos(t)\cos(\omega) - b_0\sin(t)\sin(\omega), \\ Y_0 + a_0\cos(t)\sin(\omega) + b_0\sin(t)\cos(\omega) \end{cases}$$

where (X_0, Y_0) *is the center of the ellipse, and* ω *is the angle between the* x*-axis and the major axis of the ellipse.*

```
In[8]:=inverseEllipse[a0_, b0_, X00_, Y00_, w0_, aa_, bb_,
domx_, domy_] :=
Module[{a = a0, b = b0, X0 = X00, Y0 = Y00, w = w0},
A = {X0 + a* Cos[t] Cos[w] - b *Sin[t] Sin[w],
Y0 + a* Cos[t] Sin[w] + b Sin[t] Cos[w]};
ParametricPlot[{A, inversionElliptic[A, aa, bb]} // Evaluate,
{t, 0,2Pi}, AxesLabel -> {x, y},
PlotStyle -> {{Thickness[0.006], Blue}, {Thickness[0.006], Red}},
PlotRange -> {{-domx, domx}, {-domy, domy}},
PlotStyle -> AbsoluteThickness[9],
Prolog -> {Opacity[0.15], Gray, Disk[{0, 0}, {aa, bb}]},
Epilog -> {Black, Circle[{0, 0}, {aa, bb}]}]]
In[9]:=Manipulate[inverseEllipseE[a0, b0, X0, Y0,
angle, aa, bb, domx, domy],
{{a0, 4},1, 10, 1, Appearance -> "Labeled"},
{{b0, 3}, 1, 10, 1,Appearance -> "Labeled"},
{{X0, 3}, -10, 10, 1,Appearance -> "Labeled"},
{{Y0, 3}, -10, 10, 1,Appearance -> "Labeled"},
{{angle, 0}, 0, 2 Pi, Pi/24,Appearance -> "Labeled"},
{{aa, 3}, 1, 6, 1,Appearance -> "Labeled"},
{{bb, 2}, 1, 6, 1,Appearance -> "Labeled"},
```

```
{{domx, 10}, 5, 50, 1,Appearance -> "Labeled"},
{{domy, 10}, 5, 50, 1,Appearance -> "Labeled"}]
Out[9]:= See Figure 10
```

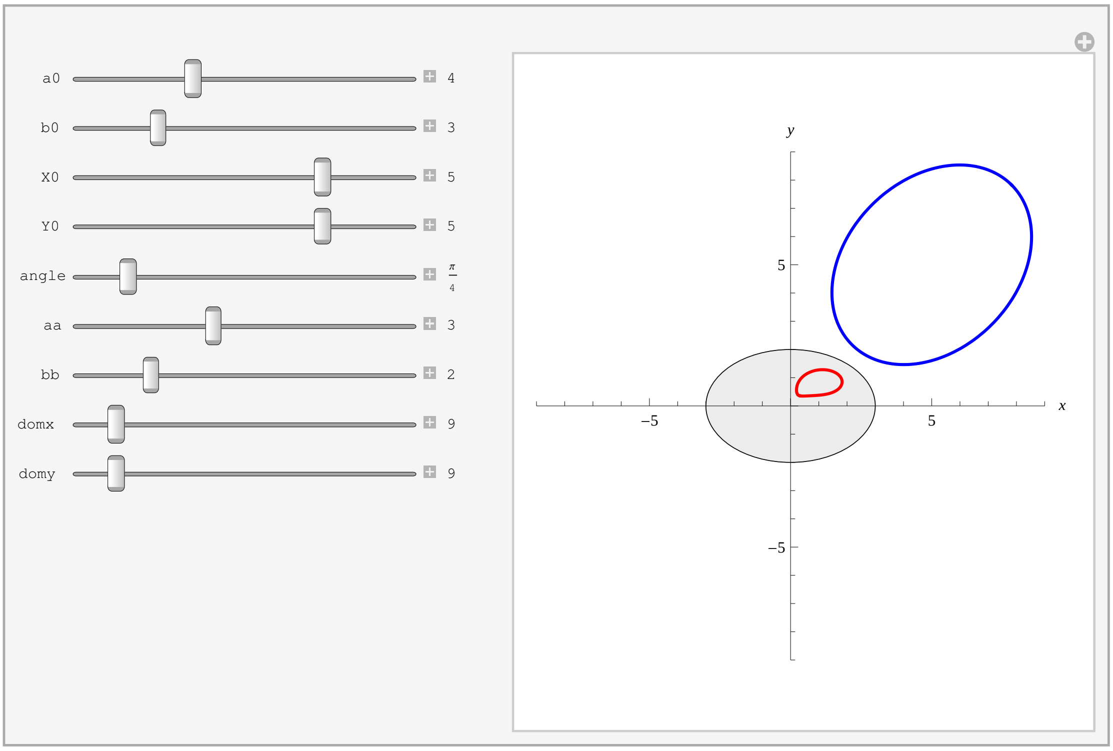


FIGURE 10. Elliptic Inversion of an Ellipse

In Table 1 we show some examples of conics and their elliptic curves respect to $\mathcal{E}_{a,b}$. The curve color blue is the curve given, and the curve of color red is the elliptic curve.

4.1. **Inversion in an ellipse of parametric curves.** Note that if $P = (f(t), g(t))$, then the inverse respect to $\mathcal{E}_{a,b}$ has equations

$$x = \frac{a^2b^2 f(t)}{b^2 f(t) + a^2 g(t)} \qquad y = \frac{a^2b^2 g(t)}{b^2 f(t) + a^2 g(t)}$$

In Tables 2 and 3 we show some examples of parametric curves and its elliptic curve respect to the ellipse $\mathcal{E}_{a,b}$.

Example 4.4. *With the following code you can generate a family of curves and their elliptic curves, see Table 4.*

```
Clear[A, B];
A = Table[{t, t^2 + k/10}, {k, -80, 20, 2}];
B = Table[inversionElliptic[{t, t^2 + k/10}, 2, 1],
{k, -80, 80, 2}];
```

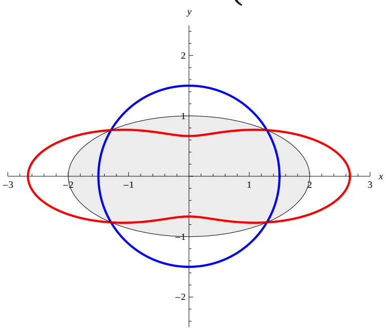
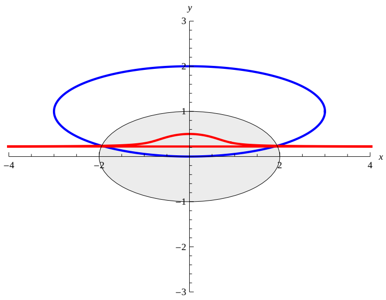
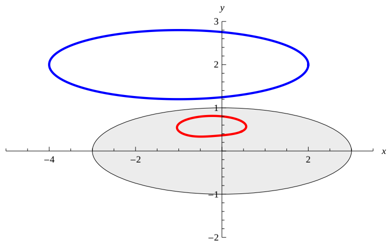
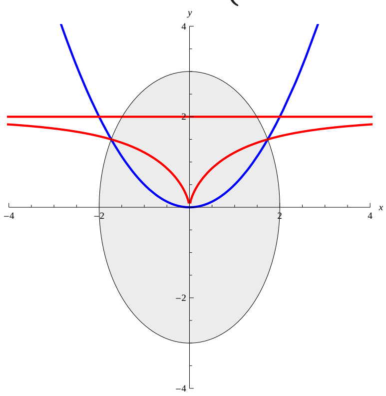
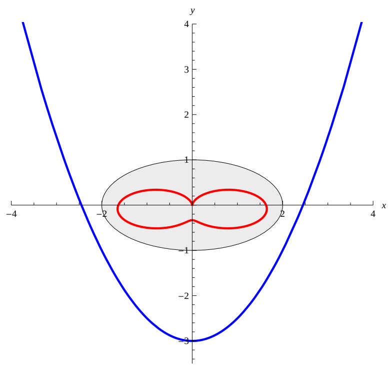
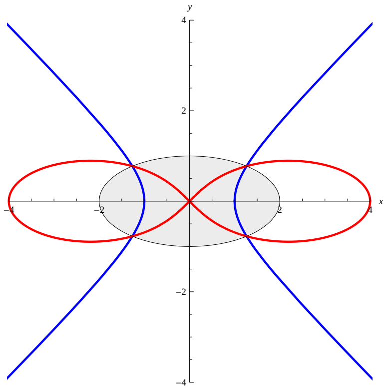
$\mathcal{E}_{2,1}$; Circumference: $\begin{cases} x = 1.5\cos t \\ y = 1.5\sin t \end{cases}$ 	$\mathcal{E}_{2,1}$; Ellipse: $\begin{cases} x = 3\cos t \\ y = \sin t + 1 \end{cases}$ 
$\mathcal{E}_{3,1}$; Ellipse: $\begin{cases} x = 3\cos t - 1 \\ y = 2 + 0.8\sin t \end{cases}$ 	$\mathcal{E}_{2,3}$; Parabola: $\begin{cases} x = 2t \\ y = 2t^2 \end{cases}$ 
$\mathcal{E}_{2,1}$; Parabola: $\begin{cases} x = 2t \\ y = 2t^2 - 3 \end{cases}$ 	$\mathcal{E}_{2,1}$; Hyperbola: $\begin{cases} x = 3\cos t \\ y = \sin t + 1 \end{cases}$ 

Table 1. Some parametric conics and their elliptic curves

```
ParametricPlot[{A} // Evaluate, {t, -5, 5},
 PlotRange -> {{-2, 2}, {-2, 2}}, PlotStyle -> Thickness[0.001],
 ImageSize -> 500, Epilog -> Circle[{0, 0}, {2, 1}]]
ParametricPlot[{B} // Evaluate, {t, -5, 5},
 PlotRange -> {{-2, 2}, {-2, 2}}, PlotStyle -> Thickness[0.001],
```

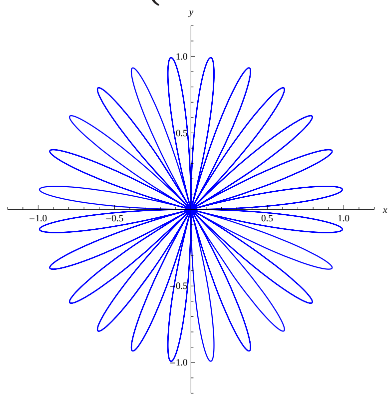
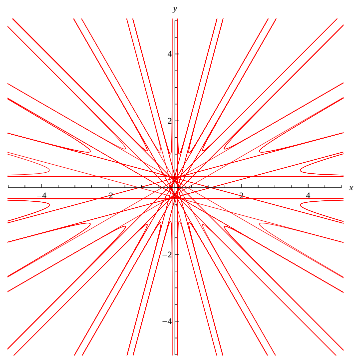
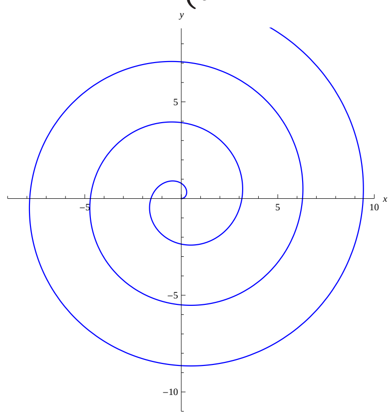
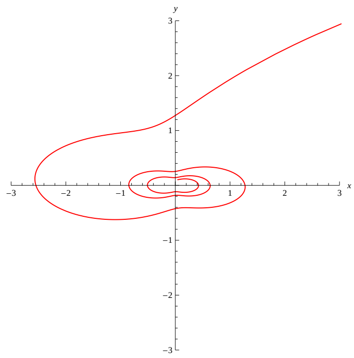
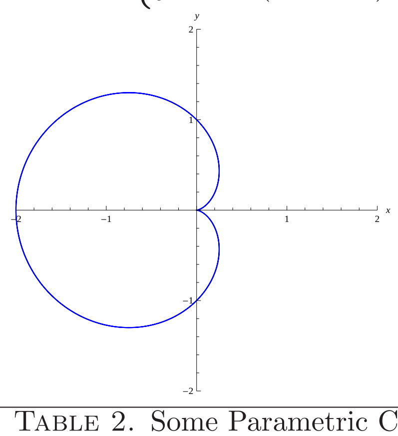
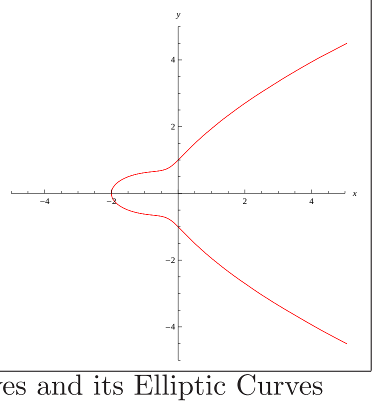
$\mathcal{E}_{2,1};\quad \Omega = \begin{cases} x = \sin(12t)\cos t \\ y = \sin(12t)\sin t \end{cases}$	$\psi(\Omega)$:
	
$\mathcal{E}_{2,1};\quad \Omega = \begin{cases} x = t\cos 2t \\ y = t\sin 2t \end{cases}$	$\psi(\Omega)$:
	
$\mathcal{E}_{2,1};\quad \Omega = \begin{cases} x = \cos t(1-\cos t) \\ y = \sin t(1-\cos t) \end{cases}$	$\psi(\Omega)$:
	

Table 2. Some Parametric Curves and its Elliptic Curves

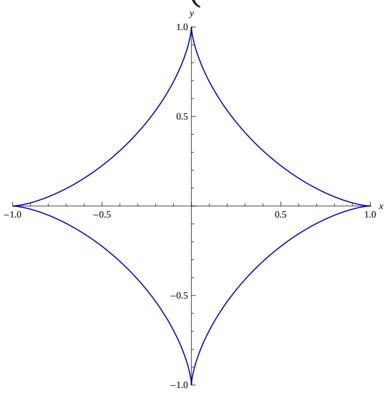
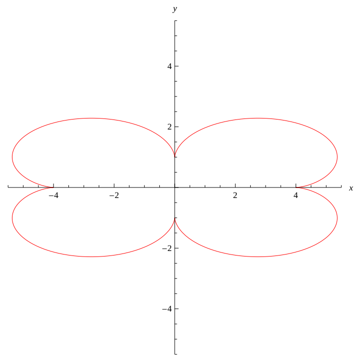
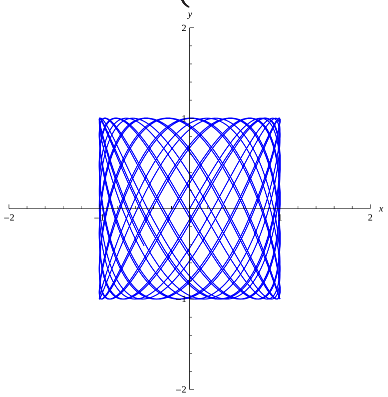
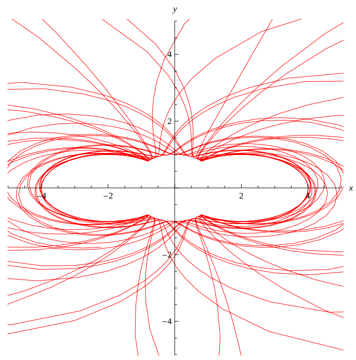
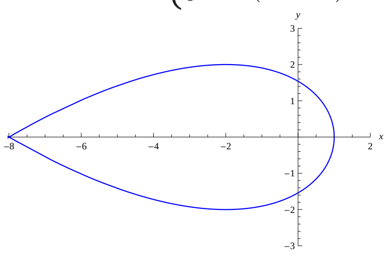
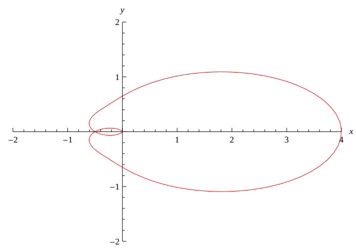
$\mathcal{E}_{2,1}$; $\Omega = \begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}$ 	$\psi(\Omega)$: 
$\mathcal{E}_{2,1}$; $\Omega = \begin{cases} x = \sin t \\ y = \sin \sqrt{3}t \end{cases}$ 	$\psi(\Omega)$: 
$\mathcal{E}_{2,1}$; $\Omega = \begin{cases} x = 1 - 3t^2 \\ y = t(3 - t^2) \end{cases}$ 	$\psi(\Omega)$: 

Table 3. Some Parametric Curves and its Elliptic Curves

```
ImageSize -> 500, Epilog -> Circle[{0, 0}, {2, 1}]]
```

5. Pappus Chain

The classical inversion has a lot of applications, such as the Pappus Chain Theorem, Feuerbach's Theorem, Steiner Porism, the problem of Apollonius, among others [2, 10, 12]. In this section, we generalize The Pappus Chain Theorem with respect to ellipses.

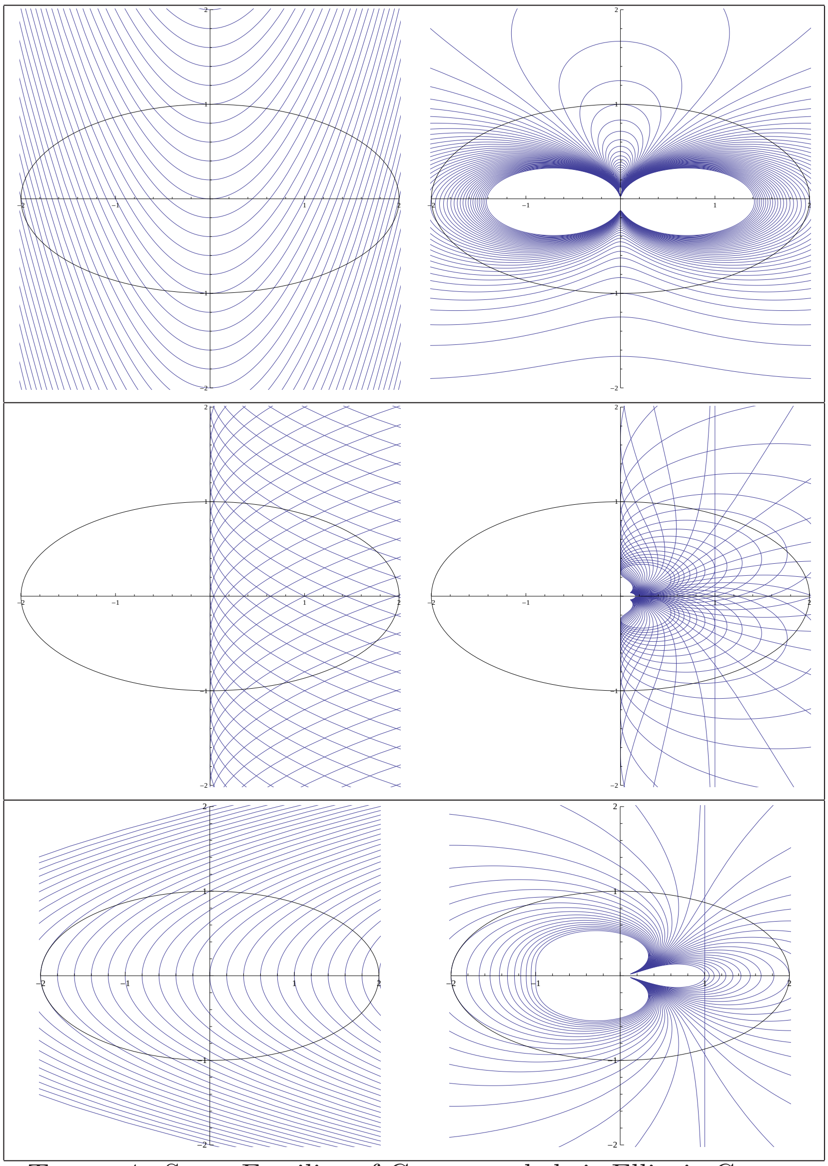


TABLE 4. Some Families of Curves and their Elliptic Curves

Theorem 5.1. *Let E be a semiellipse with principal diameter $\overline{AB}$, and E' and E_0 semiellipses on the same side of $\overline{AB}$ with principal diameters $\overline{AC}$ and $\overline{CD}$ respectively, and $E \sim E_0, E_0 \sim E'$, see Figure 11. Let $E_1, E_2, \ldots$ be a sequence of ellipses tangent to E and E', such that E_n is tangent to E_{n-1} and $E_n \sim E_{n-1}$ for all $n \geq 1$. Let r_n be the semi-minor axis of E_n and h_n the distance of the center of E_n from $\overline{AB}$. Then $h_n = 2nr_n$*

Proof. Let ψ_i the elliptic inversion such that $\psi_i(E_i) = E_i$, (in Figure 11 we select $i = 2$), i.e., the elliptic of inversion is of radius t_i and center B, where t_i is the length of the tangent segment to the Ellipse E from the point B.

By Theorem 4.2, $\psi_i(E)$ and $\psi_i(E_0)$ are perpendicular lines to the line $\overleftrightarrow{AB}$ and tangentes to the ellipse E_i. Hence, ellipses $\psi_i(E_1), \psi_i(E_2), \ldots$ will

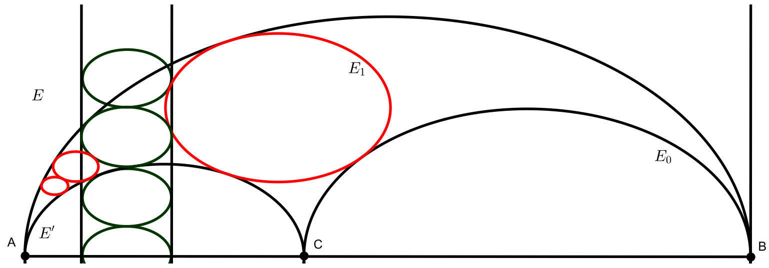


Figure 11. Elliptic Pappus Chain.

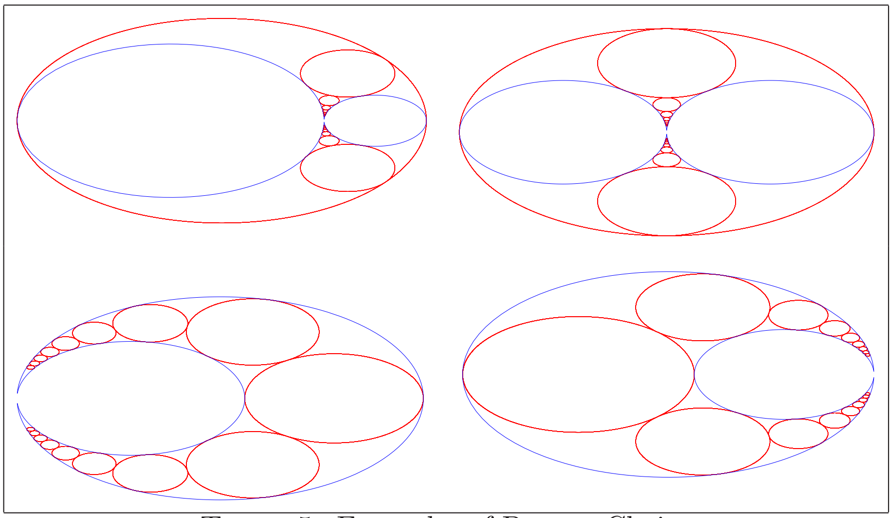


Table 5. Examples of Pappus Chains

also invert to tangent ellipses to parallel lines $\psi_i(E)$ and $\psi_i(E_0)$. Whence $h_i = 2ir_i$.

Example 5.1. *The following code generate some examples of Pappus chains, See Table 5.*

```
Clear[A, B, a, b];
a = 2; b = 1; c = 7.1;
A = Table[{c +  a*Cos[u], i + b*Sin[u]}, {i, -16, 16, 2*b}];
AA = {c - a, u}; AAA = {c + a, u};
BB = inversionElliptic[AA, a, b];
BBB = inversionElliptic[AAA, a, b];
B = Table[inversionElliptic[{c + a*Cos[u], i + b* Sin[u]}, a, b],
{i, -16, 16, 2*b}];
ParametricPlot[{B, BB, BBB} // Evaluate, {u, -100, 100},
PlotRange -> {{0, 0.8}, {-0.2, 0.2}},
PlotStyle -> {{Thickness[0.0009], Red}, {Thickness[0.001],Blue},
{Thickness[0.001], Blue}}, ImageSize -> 600, Axes -> False]
```

In Table 5 we show some examples of Pappus chains.

6. Concluding remarks

The study of elliptic inversion suggests interesting and challenging problems. For example, generalized the Steiner Porism or Apollonius Problems with respect to ellipses.

7. Acknowledgments

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