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TWO INEQUALITIES FOR A POINT IN THE PLANE OF A TRIANGLE

JIAN LIU

Abstract. In this paper we establish two new geometric inequalities involving an arbitrary point in the plane of a triangle. It is interesting that the equalities in both inequalities hold if and only if the point coincide with a special point of the original triangle. We also give a related geometric identity and several new inequalities by the main results. Finally, two elegant conjectures for the Erdös-Mordell inequality are put forward.

1. Introduction and main results

Let P be a point in the plane of triangle ABC. Denote by R_1, R_2, R_3 the distance of P from the vertices A, B, C, and r_1, r_2, r_3 the distance of P from the sidelines BC, CA, AB respectively. If P lies in the interior of ABC, then we have the following famous Erdös-Mordell inequality:

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3), \tag{1}$$

with equality holds if and only if $\triangle ABC$ is equilateral and P is its center. This inequality has attracted attention of many researchers since it first appeared in [10](see, e.g. [1]-[4], [8], [9], [11]-[14], [21]-[25]). In fact, it is well known that the stronger inequality:

$$R_1 + R_2 + R_3 \geq \left(\frac{b}{c} + \frac{c}{b}\right) r_1 + \left(\frac{c}{a} + \frac{a}{c}\right) r_2 + \left(\frac{a}{b} + \frac{b}{a}\right) r_3 \tag{2}$$

holds for interior point P, where $a = BC, b = CA, c = AB$. Equality holds if and only if P is the circumcenter of $\triangle ABC$.

In [8], N.Dergiades generalized (2) to an arbitrary point in the plane:

$$R_1 + R_2 + R_3 \geq \left(\frac{b}{c} + \frac{c}{b}\right) \vec{r_1} + \left(\frac{c}{a} + \frac{a}{c}\right) \vec{r_2} + \left(\frac{a}{b} + \frac{b}{a}\right) \vec{r_3}, \tag{3}$$

where $\vec{r_1}, \vec{r_2}, \vec{r_3}$ denote the directed (signed) distances from P to the sidelines BC, CA, AB respectively.

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The directed distance $\vec{r_1}$ from P to BC is defined as follows: When the orientation around vertices P, B, C is counterclockwise, then $\vec{r_1}$ is positive and $\vec{r_1} = r_1$; Conversely, $\vec{r_1}$ is negative and $\vec{r_1} = -r_1$ (If P lies on the line BC then $\vec{r_1} = r_1 = 0$, etc). Similarly, we define $\vec{r_2}$ and $\vec{r_3}$. For example, $\vec{r_1}, \vec{r_2}, \vec{r_3}$ are all positive in Figure 1; $\vec{r_2}$ is negative and $\vec{r_1}, \vec{r_3}$ are positive in Figure 2. In addition, we shall use directed ares of triangles. The definition of directed area of a triangle is as follows: Given a triangle XYZ, if the orientation around the vertices X, Y, Z is counterclockwise, then the directed area $\vec{S}_{\triangle XYZ}$ of $\triangle XYZ$ is positive and $\vec{S}_{\triangle XYZ} = S_{\triangle XYZ}$; Conversely, $\vec{S}_{\triangle XYZ}$ is negative and $\vec{S}_{\triangle XYZ} = -S_{\triangle XYZ}$ (If P lies on the line BC then $\vec{S}_{\triangle XYZ} = S_{\triangle XYZ} = 0$, etc). For example, we have $\vec{S}_{\triangle ABC} = S_{\triangle ABC}$ and $\vec{S}_{\triangle DEF} = -S_{\triangle DEF}$ in Figure 2.

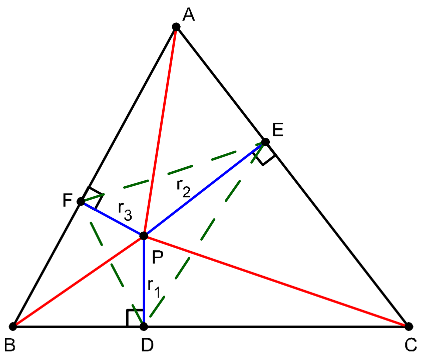

Figure 1

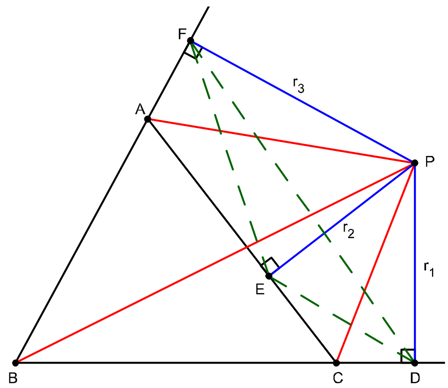

Figure 2

D.Nikolaos' proof used the following simple inequality:

$$aR_1 \geq b\vec{r_2} + c\vec{r_3}, \tag{4}$$

(two similar relations are also valid) with equality holds if and only if $PA \perp BC$.

Recently, the author first found that the following interesting inequality can be derived by using (4):

$$a^2\left(R_1^2 - r_1^2\right) + b^2\left(R_2^2 - r_2^2\right) + c^2\left(R_3^2 - r_3^2\right) \geq 4S^2, \tag{5}$$

where S is the area of $\triangle ABC$. Equality holds if and only if P is the orthocenter of $\triangle ABC$.

The proof of (5) is very simple. By (4) and the evident identity

$$a\vec{r_1} + b\vec{r_2} + c\vec{r_3} = 2\vec{S}, \tag{6}$$

where $\vec{S}$ is the directed area of $\triangle ABC$, we have

$$\begin{aligned}
&a^2\left(R_1^2 - r_1^2\right) + b^2\left(R_2^2 - r_2^2\right) + c^2\left(R_3^2 - r_3^2\right) - 4S^2 \\
&\geq (b\vec{r_2} + c\vec{r_3})^2 + (c\vec{r_3} + a\vec{r_1})^2 + (a\vec{r_1} + b\vec{r_2})^2 - (a^2r_1^2 + b^2r_2^2 + c^2r_3^2) \\
&\quad -(a\vec{r_1} + b\vec{r_2} + c\vec{r_3})^2 \\
&= a^2\vec{r_1}^2 + b^2\vec{r_2}^2 + c^2\vec{r_3}^2 - (a^2r_1^2 + b^2r_2^2 + c^2r_3^2) \\
&= 0.
\end{aligned}$$

Thus, inequality (5) is proved and the equality in (5) holds only when $PA \perp BC, PB \perp CA, PC \perp AB$, i.e., P is the orthocenter of $\triangle ABC$.

Inequality (5) inspires the author to find similar results. Finally, we obtain

Theorem 1.1. *Let P be a point in the plane of triangle ABC with sides a, b, c, the semi-perimeter s and the area S. Then*

$$a(s-a)(R_1^2 - r_1^2) + b(s-b)(R_2^2 - r_2^2) + c(s-c)(R_3^2 - r_3^2) \geq 2S^2. \tag{7}$$

Equality holds if and only if P coincide with the symmetrical point I' of the incentre I with respect to the circumcenter O of $\triangle ABC$.

If we denote by h_a, h_b, h_c the altitudes corresponding to sides a, b, c and denote by r_a, r_b, r_c the corresponding radii of the excircles of $\triangle ABC$, then inequality (7) is equivalent to

$$\frac{R_1^2 - r_1^2}{h_a r_a} + \frac{R_2^2 - r_2^2}{h_b r_b} + \frac{R_3^2 - r_3^2}{h_c r_c} \geq 1, \tag{8}$$

since we have $ah_a = 2S, (s-a)r_a = S$, etc..

Theorem 1.1 implies the following interesting conclusion: If $\triangle ABC$ is given, then the left-hand side of (7) or (8) attains the minimum value at when P coincides with the point I'.

After finding and proving inequality (7), the author noted a succinct formula about point I' (see Lemma 2.6 below). This formula motives the author to find again a new inequality whose equality condition is the same as in (7). In this respect we prove the following:

Theorem 1.2. *Let P be a point in the plane of triangle ABC with the directed area $\vec{S}$ and the circumradius R, let $\vec{S_p}$ be the directed area of the pedal triangle DEF of P with respect to triangle ABC. Then*

$$\vec{r_1} + \vec{r_2} + \vec{r_3} + \frac{2R\vec{S_p}}{\vec{S}} \leq 2R. \tag{9}$$

Equality holds if and only if P coincide with the symmetrical point I' of the incentre I with respect to the circumcenter O of $\triangle ABC$.

In particular, when P lies inside triangle ABC, it follows from (9) that

$$r_1 + r_2 + r_3 + \frac{2RS_p}{S} \leq 2R, \tag{10}$$

where S_p is the area of the pedal triangle DEF.

In [18], the author conjectured that the following inequality:

$$r_1 + r_2 + r_3 \leq 2R_p + \frac{2RS_p}{S} \tag{11}$$

holds for any interior point P of $\triangle ABC$, where R_p is circumradius of the pedal triangle DEF.

It is interesting to compare (10) with (11).

2. Proofs of Theorems

2.1. **Proof of Theorem 1.1.** In order to prove Theorem 1.1, we need the following several lemmas in which Lemma 1, 2, 3 and 4 are all well-known in geometry(cf. [5], [6], [7], [22], [26]).

Lemma 2.1. *Let P and M be two points in the plane of $\triangle ABC$ and let (x, y, z) be the barycentric coordinates of M with respect to $\triangle ABC$. Then*

$$(x+y+z)^2 PM^2 = (x+y+z)(xPA^2+yPB^2+zPC^2)-(yza^2+zxb^2+xyc^2), \tag{12}$$

where a, b, c are the lengths of the sides BC, CA, AB respectively.

In particular, putting $M = A$ in (12) we obtain the following important consequence (e.g. see [22]):

Lemma 2.2. *Let P be a point with barycentric coordinates (x, y, z) in the plane of $\triangle ABC$. Then*

$$(x+y+z)^2 PA^2 = (x+y+z)(yc^2+zb^2)-(yza^2+zxb^2+xyc^2), \tag{13}$$

where a, b, c are the lengths of the sides BC, CA, AB respectively.

For the distances from a point to the sides of a triangle, we have the following known formulae:

Lemma 2.3. *Let P be a point in the plane of $\triangle ABC$ with barycentric coordinates (x, y, z) and let $\vec{r_1}, \vec{r_2}, \vec{r_3}$ be the directed distances from P to the sidelines BC, CA, AB respectively. Then*

$$\vec{r_1} = \frac{xh_a}{x+y+z}, \ \vec{r_2} = \frac{yh_b}{x+y+z}, \ \vec{r_3} = \frac{zh_c}{x+y+z},$$

where h_a, h_b, h_c are altitudes corresponding to the sides BC, CA, AB.

Lemma 2.4. *Let P_i be points in the plane of $\triangle ABC$ with barycentric coordinates (x_i, y_i, z_i) $(i = 1, 2, 3)$. Then P_1, P_2, P_3 are collinear if and only if the following determinant holds:*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0. \tag{14}$$

Lemma 2.5. *Let $p_1, p_2, p_3, q_1, q_2, q_3$ be real numbers, then the following ternary quadratic inequality:*

$$p_1x^2 + p_2y^2 + p_3z^2 \geq q_1yz + q_2zx + q_3xy \tag{15}$$

holds for all real numbers x, y, z if and only if $p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, 4p_2p_3 - q_1^2 \geq 0, 4p_3p_1 - q_2^2 \geq 0, 4p_1p_2 - q_3^2 \geq 0$, and

$$D \equiv 4p_1p_2p_3 - (q_1q_2q_3 + p_1q_1^2 + p_2q_2^2 + p_3q_3^2) \geq 0. \tag{16}$$

The equality conditions of (15) (suppose that $p_1 > 0, p_2 > 0, p_3 > 0$) are as follows:

1° *If $D = 0$, $4p_1p_2 - q_3^2 = 0, 4p_3p_1 - q_2^2 = 0$, then $4p_2p_3 - q_1^2 = 0$ and the equality in (15) holds only when $2p_1x = q_3y + q_2z$.*

2° *If $D \geq 0, 4p_2p_3 - q_1^2 > 0, 4p_3p_1 - q_2^2 > 0, 4p_1p_2 - q_3^2 = 0$, then the equality in (15) holds only when $D = 0, z = 0, 2p_1x = q_3y$.*

3° *If $D \geq 0, 4p_2p_3 - q_1^2 > 0, 4p_3p_1 - q_2^2 > 0, 4p_1p_2 - q_3^2 > 0, x \neq 0, y \neq 0, z \neq 0$, then the equality in (15) holds only when $D = 0, (2p_1q_1 + q_2q_3)x = (2p_2q_2 + q_3q_1)y = (2p_3q_3 + q_1q_2)z$.*

The above lemma not only gives a necessary and sufficient condition of (15)(this is well-known, see e.g. [20]) but also give its equality conditions. Here, we offer an elementary proof as follows:

Proof. Inequality (15) can be rewritten as

$$p_1x^2 - (q_3y + q_2z)x + p_2y^2 - q_1yz + p_3z^2 \geq 0. \tag{17}$$

We know this inequality holds for any real number x if and only if $p_1 \geq 0$, and

$$p_2y^2 - q_1yz + p_3z^2 \geq 0, \tag{18}$$

$$(q_3y + q_2z)^2 - 4p_1(p_2y^2 - q_1yz + p_3z^2) \leqslant 0. \tag{19}$$

Again, (18) holds for any real numbers y, z if and only if $p_2 \geq 0, p_3 \geq 0, q_1^2 - 4p_2p_3 \leqslant 0$. Inequality (18) is equivalent to

$$(4p_1p_2 - q_3^2)y^2 - 2(2p_1q_1 + q_2q_3)yz + (4p_3p_1 - q_2^2)z^2 \geq 0, \tag{20}$$

which holds for any real numbers y, z if and only if $4p_1p_2 - q_3^2 \geq 0, 4p_3p_1 - q_2^2 \geq 0$, and

$$[-2(2p_1q_1 + q_2q_3)]^2 - 4(4p_1p_2 - q_3^2)(4p_3p_1 - q_2^2) \leqslant 0,$$

or

$$(4p_1p_2 - q_3^2)(4p_3p_1 - q_2^2) - (2p_1q_1 + q_2q_3)^2 \geq 0, \tag{21}$$

which is equivalent to

$$16p_1(p_1q_1^2 + p_2q_2^2 + p_3q_3^2 + q_1q_2q_3 - 4p_1p_2p_3) \leqslant 0.$$

Since $p_1 \geq 0$ we have

$$4p_1p_2p_3 - (q_1q_2q_3 + p_1q_1^2 + p_2q_2^2 + p_3q_3^2) \geq 0, \tag{22}$$

i.e. $D \geq 0$.

Combining with the above arguments, we conclude that the necessary and sufficient conditions of (15) holds for any real numbers x, y, z are $p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, 4p_2p_3 - q_1^2 \geq 0, 4p_3p_1 - q_2^2 \geq 0, 4p_1p_2 - q_3^2 \geq 0, D \geq 0$.

In what follows, we are going to discuss the equality condition in (15).

Clearly, the equality in (17) and then (15) holds if and only if

$$2p_1x - q_3y - q_2z = 0 \tag{23}$$

and there is equality in (18) or (20), i.e.

$$(4p_1p_2 - q_3^2)y^2 - 2(2p_1q_1 + q_2q_3)yz + (4p_3p_1 - q_2^2)z^2 = 0. \tag{24}$$

According to these, we will discuss the equality conditions of (15) under the different cases(suppose that $p_1 > 0, p_2 > 0, p_3 > 0$ for each case):

Case 1° $D = 0, 4p_1p_2 - q_3^2 = 0, 4p_3p_1 - q_2^2 = 0$.

In this case, we have $16p_2p_3p_1^2 = q_2^2q_3^2$ and (21) becomes an identity. Hence it follows from (21) that $2p_1q_1 + q_2q_3 = 0$. Thus(24) holds and $4p_2p_3 - q_1^2 = \dfrac{q_2^2q_3^2}{4p_1^2} - q_1^2 = 0$. Also, the equality in (15) holds only when (23) is valid.

Case 2° $D \geq 0, 4p_2p_3 - q_1^2 >, 4p_3p_1 - q_2^2 > 0, 4p_1p_2 - q_3^2 = 0$.

By $4p_1p_2 - q_3^2 = 0$ and (21), we have $2p_1q_1 + q_2q_3 = 0$. Then (24) becomes $(4p_3p_1 - q_2^2)z^2 = 0$, hence $z = 0$ and it follows from (23) that $2p_1x = q_3y$. Therefore, there is equality in (15) only when $D = 0, z = 0, 2p_1x = q_3y$.

Case 3° $D \geq 0, 4p_2p_3 - q_1^2 > 0, 4p_3p_1 - q_2^2 > 0, 4p_1p_2 - q_3^2 > 0, x \neq 0, y \neq 0, z \neq 0$.

First, it is easy to know that there is equality in (20) only when $D = 0$ and

$$2(4p_1p_2 - q_3^2)y - 2(2p_1q_1 + q_2q_3)z = 0,$$

i.e.

$$\frac{y}{z} = \frac{2p_1q_1 + q_2q_3}{4p_1p_2 - q_3^2}. \tag{25}$$

Using (23) and (25), we easily get

$$\frac{x}{z} = \frac{2p_2q_2 + q_3q_1}{4p_1p_2 - q_3^2}. \tag{26}$$

In addition, using $D = 0$, it is easy to prove the following two identities:

$$\frac{2p_1q_1 + q_2q_3}{4p_1p_2 - q_3^2} = \frac{2p_3q_3 + q_1q_2}{2p_2q_2 + q_3q_1}, \tag{27}$$

$$\frac{2p_2q_2 + q_3q_1}{4p_1p_2 - q_3^2} = \frac{2p_3q_3 + q_1q_2}{2p_1q_1 + q_2q_3}. \tag{28}$$

Thus, it follows from (25)-(28) that

$$(2p_1q_1 + q_2q_3)x = (2p_2q_2 + q_3q_1)y = (2p_3q_3 + q_1q_2)z. \tag{29}$$

Hence the equality in (15) under the third case if and only if $D = 0$ and (29) are both valid. This completes the proof of Lemma 2.5.

We now prove Theorem 1.1.

Proof. Denote by (x, y, z) the barycentric coordinates of P with respect to $\triangle ABC$. By Lemma 2.2, it is easy to obtain that

$$R_1^2 = \frac{y^2c^2 + z^2b^2 + yz(b^2 + c^2 - a^2)}{(x + y + z)^2}. \tag{30}$$

From and Lemma 2.3, we have

$$R_1^2 - r_1^2 = \frac{y^2c^2 + z^2b^2 + yz(b^2 + c^2 - a^2)}{(x + y + z)^2} - \frac{4x^2S^2}{a^2(x + y + z)^2}. \tag{31}$$

Therefore

$$\begin{aligned} &\sum a(s-a)(R_1^2 - r_1^2) \\ = \; &\frac{\sum a(s-a)[y^2c^2 + z^2b^2 + yz(b^2 + c^2 - a^2)]}{(\sum x)^2} - \frac{4S^2}{(\sum x)^2} \sum \frac{s-a}{a} x^2, \end{aligned}$$

where $\sum$ denote cyclic sums. Hence, the inequality (7) of Theorem 1.1 is equivalent to

(32)

$$\sum a(s-a)[y^2c^2 + z^2b^2 + yz(b^2 + c^2 - a^2)] - 4S^2 \sum \frac{s-a}{a} x^2 \geq 2\left(\sum x\right)^2 S^2.$$

Replacing $x \to xa, y \to yb, z \to zc$, again we know (32) is equivalent to
(33)

$$abc\sum (s-a)\left[y^2bc+z^2bc+yz(b^2+c^2-a^2)\right] -4S^2\sum a(s-a)x^2 \geq 2\left(\sum xa\right)^2 S^2.$$

Multiplying both sides of (33) by 8, then using $2s = a+b+c$ and Heron's formula:

$$16S^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c), \tag{34}$$

inequality (33) becomes

$$\begin{aligned}
&4abc\sum(b+c-a)\left[y^2bc+z^2bc+yz(b^2+c^2-a^2)\right] - \qquad (35)\\
&-(a+b+c)(b+c-a)(c+a-b)(a+b-c)\cdot \left[\sum a(b+c-a)x^2+\left(\sum xa\right)^2\right] \geq 0.
\end{aligned}$$

Expansion and simplification give

$$p_1x^2+p_2y^2+p_3z^2-(q_1yz+q_2zx+q_3xy) \geq 0, \tag{36}$$

where

$$\begin{aligned}
p_1 &= a\left[(b+c)a^4-2(b+c)(b-c)^2a^2+4abc(b-c)^2+(b-c)^2(b+c)^3\right],\\
p_2 &= b\left[(c+a)b^4-2(c+a)(c-a)^2b^2+4abc(c-a)^2+(c-a)^2(c+a)^3\right],\\
p_3 &= c\left[(a+b)c^4-2(a+b)(a-b)^2c^2+4abc(a-b)^2+(a-b)^2(a+b)^3\right],\\
q_1 &= 2bc(b+c-a)\\
&\quad\cdot\left[3a^3+(b+c)a^2+a(2bc-3b^2-3c^2)-(b+c)(b-c)^2\right],\\
q_2 &= 2ca(c+a-b)\\
&\quad\cdot\left[3b^3+(c+a)b^2+b(2ca-3c^2-3a^2)-(c+a)(c-a)^2\right],\\
q_3 &= 2ab(a+b-c)\\
&\quad\cdot\left[3c^3+(a+b)c^2+c(2ab-3a^2-3b^2)-(a+b)(a-b)^2\right].
\end{aligned}$$

We now apply Lemma 2.5 to prove inequality (36). First, with the help of the mathematics software for calculating, we can check the following identity:

$$4p_1p_2p_3-q_1q_2q_3-p_1q_1^2-p_2q_2^2-p_3q_3^2=0. \tag{37}$$

By Lemma 2.5, it remains to prove that $p_1 > 0, 4p_2p_3-q_1^2 \geq 0$ and their analogues. Because of symmetry, we only need to prove these two inequalities.

To prove $p_1 > 0$ we need to prove that

$$Q_0 \equiv (b+c)a^4-2(b+c)(b-c)^2a^2+4abc(b-c)^2+(b-c)^2(b+c)^3 > 0. \tag{38}$$

Putting $s-a = u, s-b = v, s-c = w$, then $a = v+w, b = w+u, c = u+v$. Furthermore, we can check that

$$\begin{aligned}
Q_0 \equiv\ & 8(v-w)^2u^3+16(v+w)(v-w)^2u^2+8(v^2-vw+w^2)(v+w)^2u\\
&+8vw(v+w)(v^2+w^2). \qquad (39)
\end{aligned}$$

Since $u > 0, v > 0, w > 0$ and $v^2-vw+w^2 > 0$, inequality $Q_0 > 0$ is valid and $p_1 > 0$ is proved.

It remains to prove that $4p_2p_3-q_1^2 \geq 0$. By calculating we get the following identities:

$$4p_2p_3 - q_1^2 = 4abcm_1k_1^2, \tag{40}$$
$$4p_3p_1 - q_2^2 = 4abcm_2k_2^2, \tag{41}$$
$$4p_1p_2 - q_3^2 = 4abcm_3k_3^2, \tag{42}$$

where

$$m_1 = a^3 - (b+c)a^2 - (b^2+c^2-6bc)a + (b+c)(b-c)^2, \tag{43}$$
$$m_2 = b^3 - (c+a)b^2 - (c^2+a^2-6ca)b + (c+a)(c-a)^2, \tag{44}$$
$$m_3 = c^3 - (a+b)c^2 - (a^2+b^2-6ab)c + (a+b)(a-b)^2, \tag{45}$$
$$k_1 = a^3 + (b+c)a^2 - a(b+c)^2 - (b+c)(b-c)^2, \tag{46}$$
$$k_2 = b^3 + (c+a)b^2 - b(c+a)^2 - (c+a)(c-a)^2, \tag{47}$$
$$k_3 = c^3 + (a+b)c^2 - c(a+b)^2 - (a+b)(a-b)^2. \tag{48}$$

To show that $4p_2p_3 - q_1^2 \geq 0$, we have to prove $m_1 > 0$. But m_1 can be rewritten as

$$m_1 = 4(v+w)u^2 + 4(v^2+w^2)u + 4vw(v+w),$$

where $u = s-a > 0, v = s-b > 0, w = s-c > 0$. So $m_1 > 0$ is true and $4p_2p_3 - q_1^2 \geq 0$ is proved. This completes the proof of inequality (36), and then (33), (32), (7) are proved.

Next, we shall discuss the equality conditions in (7).

Clearly, the values of k_1, k_2, k_3 may be zero. In fact, it is easy to prove that one of k_1, k_2, k_3 at most equal zero(we omit details). Hence by Lemma 2.5, we will consider two cases below to discuss the equality condition in (36) and further that of (7).

Case 1. None of k_1, k_2, k_3 equals zero.

First, we will prove that equality in (7) holds only when the barycentric coordinates of P is (ak_1, bk_2, ck_3). Then we will show that this point just is the symmetrical point I' of the incentre I with respect to the circumcenter O of $\triangle ABC$.

By the hypothesis, three strict inequalities $4p_2p_3 - q_1^2 > 0, 4p_3p_1 - q_2^2 > 0, 4p_1p_2 - q_3^2 > 0$ follow from (40), (41), (42) respectively. In addition, we have

$$2p_1q_1 + q_2q_3 = 4abcu_1v_1w_1, \tag{49}$$
$$2p_2q_2 + q_3q_1 = 4abcu_2v_2w_2, \tag{50}$$
$$2p_3q_3 + q_1q_2 = 4abcu_3v_3w_3, \tag{51}$$

where

$$\begin{aligned}
u_1 &= a^3-(b+c)a^2-a(b^2+c^2-6bc)+(b+c)(b-c)^2,\\
u_2 &= b^3-(c+a)b^2-b(c^2+a^2-6ca)+(c+a)(c-a)^2,\\
u_3 &= c^3-(a+b)c^2-c(a^2+b^2-6ab)+(a+b)(a-b)^2,\\
v_1 &= b^3+(c-a)b^2-b(a-c)^2+(a-c)(a+c)^2,\\
v_2 &= c^3+(a-b)c^2-c(b-a)^2+(b-a)(b+a)^2,\\
v_3 &= a^3+(b-c)a^2-a(b-c)^2+(c-b)(c+b)^2,\\
w_1 &= c^3+(b-a)c^2-c(a-b)^2+(a-b)(a+b)^2,\\
w_2 &= a^3+(c-b)a^2-a(b-c)^2+(b-c)(b+c)^2,\\
w_3 &= b^3+(a-c)b^2-b(a-c)^2+(c-a)(c+a)^2.
\end{aligned}$$

According to Lemma 2.5, we know that the equalities of (36) and its equivalent form (33) hold if and only if $xu_1v_1w_1 = yu_2v_2w_2 = zu_3v_3w_3$. Further, the equality in (32) holds if and only if

$$\frac{x}{a}u_1v_1w_1 = \frac{y}{b}u_2v_2w_2 = \frac{z}{c}u_3v_3w_3.$$

Then by Lemma 2.3, the equality in (7) holds if and only if P coincide with the point I' whose barycentric coordinates is $\left(\frac{a}{u_1v_1w_1}, \frac{b}{u_2v_2w_2}, \frac{c}{u_3v_3w_3}\right)$. This barycentric coordinates is much complicated. In fact it can be expressed in a simple way. The author finds that the following continued equality:

$$u_1v_1w_1k_1 = u_2v_2w_2k_2 = u_3v_3w_3k_3 \tag{52}$$

holds. It can easily be checked by using the mathematics software. Therefore, the barycentric coordinates of I' can also be expressed by (ak_1, bk_2, ck_3).

Now, we will prove that I' is the symmetrical point I' of the incentre I with respect to the circumcenter O of $\triangle ABC$. We first prove three points I, O, I' are collinear. As it is well known, the barycentric coordinates of the incentre I and the circumentre O are (a, b, c) and

$$\left(a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2)\right)$$

respectively. Hence according to Lemma 2.4, to prove I, O, I' to be collinear, we need to prove that

$$\begin{vmatrix} a & b & c \\ a^2(b^2+c^2-a^2) & b^2(c^2+a^2-b^2) & c^2(a^2+b^2-c^2) \\ ak_1 & bk_2 & ck_3 \end{vmatrix} = 0.$$

Again, by the properties of the determinant, the proof can be turned into

$$\begin{vmatrix} 1 & 1 & 1 \\ k_1 & k_2 & k_3 \\ a(b^2+c^2-a^2) & b(c^2+a^2-b^2) & c(a^2+b^2-c^2) \end{vmatrix} = 0,$$

Expending gives

$$\begin{aligned}
&k_1\left[b(c^2+a^2-b^2)-c(a^2+b^2-c^2)\right]\\
&+k_2\left[c(a^2+b^2-c^2)-a(b^2+c^2-a^2)\right]\\
&+k_3\left[a(b^2+c^2-a^2)-b(c^2+a^2-b^2)\right]=0.
\end{aligned} \tag{53}$$

Putting (46)-(48) into the left of (53), one can easily check that (53) is true. Hence we proved that I, O, I' are collinear.

Successively, we will prove that I' is the the symmetrical point of I with respect to the circumenter O.

In Lemma 2.1, for $P = O$, then we get the known formula:

$$OM^2 = R^2 - \frac{yza^2 + zxb^2 + xyc^2}{(x+y+z)^2}. \tag{54}$$

Letting $M = I'$ in the above, we obtain that

$$OI'^2 = R^2 - \frac{abc(ak_2k_3 + bk_3k_1 + ck_1k_2)}{(ak_1 + bk_2 + ck_3)^2}. \tag{55}$$

By (46), (47) and (43), it is easy to get two identities:

$$ak_1 + bk_2 + ck_3 = -(a+b+c)(b+c-a)(c+a-b)(a+b-c), \tag{56}$$

$$\begin{aligned} & ak_2k_3 + bk_3k_1 + ck_1k_2 \\ = {} & (a+b+c)(b+c-a)^2(c+a-b)^2(a+b-c)^2, \end{aligned} \tag{57}$$

so that

$$OI'^2 = R^2 - \frac{abc}{a+b+c}. \tag{58}$$

Since $a + b + c = 2s$ and $abc = 4Rrs$ (r is the inradius of $\triangle ABC$), we further get $OI'^2 = R^2 - 2Rr$ which is the same as the famous Euler formula $OI^2 = R^2 - 2Rr$ (see e.g. [22, page 279]). Thus we have $OI' = OI$. Since also we have proved that I', O, I are collinear before, the point I' therefore must be the symmetrical point of the incentre I with respect to the circumcenter O.

Case 2. One of k_1, k_2, k_3 equals zero.

There is no harm in assuming that $k_1 = 0$. In this case, $4p_2p_3 - q_1^2 = 0$ follows from (40). Thus by Lemma 2.5, we know that the equality in (36) holds if and only if $x = 0, 2p_2y = q_1z$. Further, the equality in (32) holds only when

$$x = 0,\ 2p_2\frac{y}{b} = q_1\frac{z}{c},$$

i.e. the barycentric coordinates of P is $(0, bq_1, 2p_2c)$. Indeed, this point just is the point with barycentric coordinates $(0, bk_2, ck_3)$. Due to the homogeneity of barycentric coordinates (cf. [26]) it will be sufficient to prove that $q_1 : k_2 = 2p_2 : k_3$, i.e.

$$2p_2k_2 - q_1k_3 = 0. \tag{59}$$

It is easy to check that $2p_2k_2 - q_1k_3 = -2ab(a+b-c)[a^3 + (3c-b)a^2 - a(b+c)^2 + (b-c)(3c+b)(b+c)]k_1$, which shows obviously that (59) holds true when $k_1 = 0$. Again, from the proof of Case 1, we see that P is still the symmetrical point I' of the incentre I with respect to the circumcenter O if $k_1 = 0$. Therefore, the equality in (7) under the second case holds only when $P = I'$.

The point I' does not lie on the boundary of $\triangle ABC$ under Case 1 (since $k_1k_2k_3 \neq 0$). The point I' lies the boundary (except the vertices) of $\triangle ABC$ under Case 2. Combing with the arguments of Case 1 and Case 2, we know that the statements for the equality in Theorem 1.1 under these two cases are right. The proof of Theorem 1.1 is completed.

2.2. **Proof of Theorem 1.2.**

Lemma 2.6. *Let I be the incentre of $\triangle ABC$ and let O be its circumcenter. The symmetrical point of I with respect to O is I'. Then the distance between I' and the vertices A is given by*

$$I'A = 2R\sqrt{1-\sin B\sin C}, \tag{60}$$

where R is the circumradius of $\triangle ABC$ and B, C are the angles of $\triangle ABC$.

Proof. In the proof of Theorem 1.1, we have known that the barycentric coordinates of I' is (ak_1, bk_2, ck_3) (where the values of k_1, k_2, k_3 are the same as in (46), (47), (48)). Hence by Lemma 2.2 we have that

$$I'A^2 = \frac{bc(k_2c+k_3b)}{ak_1+bk_2+ck_3} - \frac{abc(ak_2k_3+bk_3k_1+ck_1k_2)}{(ak_1+bk_2+ck_3)^2}. \tag{61}$$

Putting k_2, k_3 into (61) and using (56), (57), we further get

$$I'A^2 = \frac{bc(4bca^2+a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2)}{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}. \tag{62}$$

Using the equivalent form of Heron's formula:

$$16S^2 = 2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4, \tag{63}$$

it follows that

$$\begin{aligned} I'A^2 &= \frac{bc(16aSR-16S^2)}{16S^2} = \frac{(aR-S)bc}{S} = \frac{bc(2aR-ah_a)}{2S} \\ &= \frac{abc(2R-h_a)}{2S} = 2R(2R-h_a) = 2R(2R-2R\sin B\sin C) \\ &= 4R^2(1-\sin B\sin C). \end{aligned}$$

Hence $I'A = 2R\sqrt{1-\sin B\sin C}$ is valid. This completes the proof of Lemma 2.6.

As a straightforward important consequence of Lemma 2.1, we have that

Lemma 2.7. *For any point P in the plane of $\triangle ABC$ and all real numbers x, y, z,*

$$(x+y+z)(xPA^2+yPB^2+zPC^2) \geq yza^2+zxb^2+xyc^2, \tag{64}$$

with equality if and only if $x:y:z = \vec{S}_{\triangle PBC} : \vec{S}_{\triangle PCA} : \vec{S}_{\triangle PAB}$.

The inequality (64) is called "The polar moment of the inertia inequality of Klamkin". This is one of the most important results for triangle geometric inequalities. A number of triangle inequalities can be derived from it (see e.g. [15], [16], [22])

We now prove Theorem 1.2.

Proof. Clearly, we can assume that the orientation of $\triangle ABC$ is counter-clockwise (see figure 1), then $\vec{S} = S$ and we have to prove

$$\vec{r_1}+\vec{r_2}+\vec{r_3}+\frac{2R\vec{S_p}}{S} \leq 2R. \tag{65}$$

By Lemma 2.6, Lemma 2.7 and the sine rule, we obviously get the following weighted trigonometric inequality:

$$(66)\ (x+y+z)\left[x(1-\sin B\sin C)+y(1-\sin C\sin A)+z(1-\sin A\sin B)\right] \geq\ yz\sin^2 A+zx\sin^2 B+xy\sin^2 C,$$

with equality if and only if $x:y:z=ak_1:bk_2:ck_3$ (k_1,k_2,k_3 are the same as in (46), (47), (48), respectively). That is

$$(x+y+z)^2\geq(x+y+z)(x\sin B\sin C+y\sin C\sin A+z\sin A\sin B)+ yz\sin^2 A+zx\sin^2 B+xy\sin^2 C.$$

Making substitutions $x\to x\sin A, y\to y\sin B, z\to z\sin C$, we get

$$(x\sin A+y\sin B+z\sin C)^2\geq\sin A\sin B\sin C[(x+y+z)(x\sin A +y\sin B+z\sin C)+yz\sin A+zx\sin B+xy\sin C].$$

Multiplying both sides by $4R^2$ and using the law of sines and the known formula $S=2R^2\sin A\sin B\sin C$, we obtain the equivalent inequality:

$$(67)\quad (xa+yb+zc)^2\geq\frac{S}{R}[(x+y+z)(xa+yb+zc)+yza+zxb+xyc],$$

with equality if and only if $x:y:z=k_1:k_2:k_3$.

If we put $x=\vec{r_1}, y=\vec{r_2}, z=\vec{r_3}$ in (67), then using the identities $a\vec{r_1}+b\vec{r_2}+c\vec{r_3}=2S$ (by (6) and hypothesis) and

$$(68)\qquad a\vec{r_2}\vec{r_3}+b\vec{r_3}\vec{r_1}+c\vec{r_1}\vec{r_2}=4R\vec{S_p},$$

we get

$$4S^2\geq\frac{S}{R}[2S(\vec{r_1}+\vec{r_2}+\vec{r_3})+4R\vec{S_p}].$$

Hence

$$2SR\geq S(\vec{r_1}+\vec{r_2}+\vec{r_3})+2R\vec{S_p},$$

and a division by S produces inequality (65). By virtue of the equality condition of (67), it is seen that the equality in (65) holds if and only if the the barycentric coordinates of P is (ak_1,bk_2,ck_3), i.e. P coincides with point I'. The proof of Theorem 1.2 is complete.

3. Some remarks

Remark 3.1. *We have the following inequality similar to (8):*

$$(69)\quad \frac{R_2^2+R_3^2-r_2^2-r_3^2}{r_br_c}+\frac{R_3^2+R_1^2-r_3^2-r_1^2}{r_cr_a}+\frac{R_1^2+R_2^2-r_1^2-r_2^3}{r_ar_b}\geq 2,$$

which is equivalent with

$$(70)\quad \frac{R_2^2+R_3^2-r_2^2-r_3^2}{s-a}+\frac{R_3^2+R_1^2-r_3^2-r_1^2}{s-b}+\frac{R_1^2+R_2^2-r_1^2-r_2^3}{s-c}\geq 2s.$$

The equality in (69) or (70) is the same as in (7). In fact, by using Lemma 2.2 and Lemma 2.3 we can prove the following geometric identity (we omit details here):

$$(71)\qquad \sum\frac{R_1^2-r_1^2}{h_ar_a}=\frac{1}{2}\sum\frac{R_2^2+R_3^2-r_2^2-r_3^2}{r_br_c},$$

where $\sum$ denote the circle sum. Therefore, (69) can be obtained by (8) and (71).

Remark 3.2. *The point I' in Theorem 1.1 or Theorem 1.2 may be either in the interior (including the boundaries, except the vertexes) of $\triangle ABC$ or outside the triangle. We have found some properties about point I'. For example, four points I', N, H, I form a parallelogram, where N is the Nagel point, H the orthocenter and I the incenter of $\triangle ABC$ respectively (see Figure 3).*

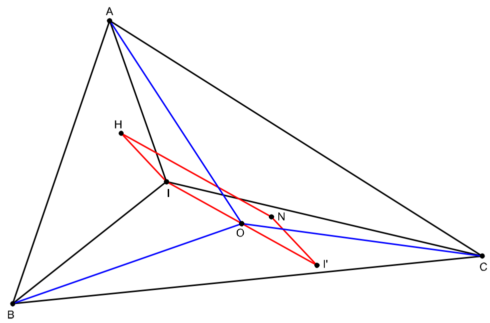

Figure 3

Remark 3.3. *Actually, the inequality (7) of Theorem 1.1 is a generalization of Heron's formula. In fact, by using (31) we can prove the following equality:*

$$R_1^2 - r_1^2 = s^2 - 2bc \tag{72}$$

holds when $P = I'$. Thus by Theorem 1.1 we get the identity:

$$a(s-a)(s^2-2bc) + b(s-b)(s^2-2ca) + c(s-c)(s^2-2ab) = 2S^2.$$

Further, it is easy to obtain the Heron's formula:

$$S = \sqrt{s(s-a)(s-b)(s-c)}. \tag{73}$$

In addition, when $P = I'$ we have the following equality similar to (72):

$$R_2^2 + R_3^2 - r_2^2 - r_3^2 = 2(s-a)^2 \tag{74}$$

which evidently shows the equality condition in (70) is right.

Remark 3.4. *The inequality (9) of Theorem 1.2 actually is equivalent to the weighted inequality (67) and the later can also be proved by Lemma 2.5. In addition, by inequality (9) and the known inequality (see [17]):*

$$\frac{S_p}{r_p} \geq \frac{S}{R}, \tag{75}$$

where r_p is the inradious of the pedal triangle DEF of interior point P with respect to triangle ABC and S_p is its area, it is seen that the beautiful linear inequality

$$r_1 + r_2 + r_3 + 2r_p \leq 2R \tag{76}$$

holds for any interior point P of $\triangle ABC$.

Remark 3.5. *From inequality (9) and the known inequality used in [19] recently:*

$$\frac{2RS_p}{S} \geq \frac{r_1r_2r_3}{r^2}, \tag{77}$$

we can get the following inequality (for interior point P):

$$r_1 + r_2 + r_3 + \frac{r_1r_2r_3}{r^2} \leq 2R, \tag{78}$$

which does not discriminate strength or weakness with (76).

Remark 3.6. *If we apply geometrical transformations to Theorem 1.1 or Theorem 1.2 or their consequences, one can obtain some new geometric inequalities. For example, applying the isogonal transformation (see e.g.[22], [23]) to inequality (8), we get*

(79)

$$a(s-a)(R_1^2r_1^2-r_2^2r_3^2)+b(s-b)(R_2^2r_2^2-r_3^2r_1^2)+c(s-c)(R_3^2r_3^2-r_1^2r_2^2) \geq 8R^2S_p^2,$$

which holds for any point P in the plane. Applying inequality (10) and other geometrical transformations, we can obtain the following inequalities (holds for any interior point P of triangle ABC):

$$\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{2R_p} \leq \frac{R_1R_2R_3}{2r_1r_2r_3R}, \tag{80}$$

$$\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c} + \frac{1}{R} \leq \frac{S}{RS_p}, \tag{81}$$

$$\frac{R_a + R_b + R_c + R}{R_p} \leq \frac{R_1R_2R_3}{r_1r_2r_3}, \tag{82}$$

where R_a, R_b, R_c are the circumradius of $\triangle PBC, \triangle PCA, \triangle PAB$ respectively.

4. **Two conjectures**

In [14], we have posed some conjectures for the Erdös-Mordell inequality. Here, we present two related new conjecture again.

Conjecture 4.1. *For any interior point of $\triangle ABC$, we have*

$$\frac{R_1 + R_2 + R_3}{r_1 + r_2 + r_3} \leq \frac{R}{2r_p}. \tag{83}$$

Conjecture 4.2. *For any interior point of $\triangle ABC$, we have*

$$\frac{R_1 + R_2 + R_3}{r_1 + r_2 + r_3} \leq \frac{2R + R_p}{2r + r_p}. \tag{84}$$

In passing, we have known that there is no comparison between (83) and (84).

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EAST CHINA JIAOTONG UNIVERSITY
JIANGXI PROVINCE NANCHANG CITY, 330013, CHINA
E-mail address: China99jian@163.com