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BISECTOR CURVES OF PLANAR RATIONAL CURVES IN LORENTZIAN PLANE

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Abstract. In this paper, the bisector curves of two planar rational curves are studied in Lorentzian plane. The bisector curves are obtained by two different methods. Consequently, some experimental results are demonstrated.

1. Introduction

The bisector for two objects is defined as the set of points equidistant from the two objects. The construction of bisectors plays an important role in many geometric computations, such as Voronoi diagrams construction, medial axis transformation, shape decomposition, mesh generation, collision-avoidance motion planning, and NC tool path generation, to mention a few.

Hofmann and Vermeer [8] formulate the bisector curve in terms of a system of polynomial equations using some auxiliary variables. By eliminating these auxiliary variables, an implicit equation of the bisector curve can be obtained. However, the process of variable elimination is slow and generates an algebraic bisector curve of high degree. Hoffmann [7] also suggests using a dimensionality paradigm in which each polynomial equation is geometrically represented as a hypersurface. The bisector curve of two planar rational curves can be computed via the intersection of three hypersurfaces in $\mathbb{R}^4$. The multiple intersection in $\mathbb{R}^4$ is inefficient, compared with that of intersecting two surfaces in $\mathbb{R}^3$. Choi transforms the problem of computing a bisector curve into that of intersecting two developable surfaces in $\mathbb{R}^3$ [2].

Farouki and Johnstone show that the bisector of a point and a rational curve in the same plane is a rational curve [4]. Given two planar rational curves, Farouki and Johnstone interpret the bisector as the envelope curve of a one-parameter family of rational point/curve bisectors [5]. The curve/curve bisector is non-rational, in general. Therefore, Farouki and Johnstone approximate the bisector curve with a sequence of discrete points [5].

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Farouki and Ramamurthy [6] develop a more precise algorithm that approximates the bisector curve with a sequence of curve segments. The approximation error can be reduced within an arbitrary bound by adaptive refinement. Elber and Kim suggest a new (symbolic) representation scheme for planar bisector curves, which allows an efficient and robust implementation of a bisector curve construction based on B-spline subdivision techniques by using bisector of two planar curves as a planar curve in $\mathbb{R}^3$ [3].

In this paper, we show that the bisector of two planar curves is also a planar curve in Lorentzian plane. We suggest the representation scheme for planar bisector curves in Lorentzian plane by using the method given in [3]. Also, the interesting examples for planar bisector curves are presented in Lorentzian plane.

2. Preliminaries

Let $\mathbb{L}^2$ be a Lorentzian plane with the Lorentzian inner product $<,>$ given by

$$< \mathbf{x}, \mathbf{y} >= x_1 y_1 - x_2 y_2, \tag{1}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2) \in \mathbb{L}^2$.

A vector $\mathbf{x}$ is said to be timelike if $< \mathbf{x}, \mathbf{x} >< 0$, spacelike if $< \mathbf{x}, \mathbf{x} >> 0$ or $\mathbf{x} = 0$, and lightlike (or null) if $< \mathbf{x}, \mathbf{x} >= 0$ and $\mathbf{x} \neq 0$. The norm of $\mathbf{x}$ is defined by

$$\|\mathbf{x}\| = \sqrt{|< \mathbf{x}, \mathbf{x} >|}. \tag{2}$$

An arbitrary curve $c(t)$ in $\mathbb{L}^2$ can locally be spacelike, timelike or null if all of its velocity vectors $\frac{dc(t)}{dt}$ are spacelike, timelike or null, respectively. More information about both the Lorentzian plane and space can be found in [1, 9, 10, 11, 12].

3. Bivariate Functions

Let $C_1(t)$ and $C_2(r)$ be regular parametric C^1-continuous plane curves in $\mathbb{R}^2_1$ given by parametrization

$$\begin{aligned} C_1(t) &= (x_1(t), y_1(t)) \\ C_2(r) &= (x_2(r), y_2(r)). \end{aligned} \tag{3}$$

We compute a bivariate polynomial function $F(t, r) = 0$ which corresponds to the bisector curve of $C_1(t)$ and $C_2(r)$. In this paper, the bivariate functions $\mathbb{F}_i(t, r)(i = 1, 2)$ are computed by two different methods in Lorentzian plane [3].

3.1. **Function $\mathbb{F}_1(t, r)$.** The tangent vectors of $C_1(s)$ and $C_2(r)$ are obtained by

$$\begin{aligned} T_1(t) &= (x_1'(t), y_1'(t)) \\ T_2(r) &= (x_2'(r), y_2'(r)). \end{aligned} \tag{4}$$

Thus, the unnormalized normal fields of $C_1(t)$ and $C_2(r)$ are obtained by

$$\begin{aligned} N_1(t) &= (y_1'(t), x_1'(t)) \\ N_2(r) &= (y_2'(r), x_2'(r)). \end{aligned} \tag{5}$$

On the other hand, the intersection point of the normal lines of $C_1(t)$ and $C_2(r)$ is given as follows:

$$C_1(t) + N_1(t)\alpha = C_2(r) + N_2(r)\beta \tag{6}$$

for some α and β.

Substituting (3) and (5) into (6) gives the following equation

$$(x_1(t), y_1(t)) + (y_1'(t), x_1'(t))\alpha = (x_2(r), y_2(r)) + (y_2'(r), x_2'(r))\beta.$$

Simple calculation implies that we have the following two equations in two unknowns α and β

$$\begin{aligned} y_1'(t)\alpha - y_2'(r)\beta &= x_2(r) - x_1(t) \\ x_1'(t)\alpha - x_2'(r)\beta &= y_2(r) - y_1(t). \end{aligned} \tag{7}$$

By using Cramer's rule in (7), we obtain the solutions of α and β as bivariate rational functions:

$$\begin{aligned} \alpha = \alpha(t,r) &= \frac{\begin{vmatrix} x_2(r) - x_1(t) & -y_2'(r) \\ y_2(r) - y_1(t) & -x_2'(r) \end{vmatrix}}{\begin{vmatrix} y_1'(t) & -y_2'(r) \\ x_1'(t) & -x_2'(r) \end{vmatrix}}, \\ \beta = \beta(t,r) &= \frac{\begin{vmatrix} y_1'(t) & x_2(r) - x_1(t) \\ x_1'(t) & y_2(r) - y_1(t) \end{vmatrix}}{\begin{vmatrix} y_1'(t) & -y_2'(r) \\ x_1'(t) & -x_2'(r) \end{vmatrix}}. \end{aligned} \tag{8}$$

The two bivariate functions $\alpha(t,r)$ and $\beta(t,r)$ provide a bivariate parametrization of the intersection point as follows:

$$P(t,r) = C_1(t) + N_1(t)\alpha(t,r) = C_2(r) + N_2(r)\beta(t,r). \tag{9}$$

In addition, since $P(t,r)$ must also be at an equal distance from $C_1(t)$ and $C_2(r)$, we have

$$\|P(t,r) - C_1(t)\| = \|P(t,r) - C_2(r)\|.$$

Using (9) and (1) implies that

$$\left|N_1^2(t)\alpha(t,r)^2\right| = \left|N_2^2(r)\beta(t,r)^2\right|. \tag{10}$$

By solving an absolute value equation in (10), we have two cases to consider. In order to represent both two cases in (10) we use $\varepsilon = \pm 1$, then we have

$$N_1^2(t)\alpha(t,r)^2 = \varepsilon N_2^2(r)\beta(t,r)^2. \tag{11}$$

Substituting (8) into (11), we get

$$N_1^2 \frac{\begin{vmatrix} x_2(r) - x_1(t) & -y_2'(r) \\ y_2(r) - y_1(t) & -x_2'(r) \end{vmatrix}^2}{\begin{vmatrix} y_1'(t) & -y_2'(r) \\ x_1'(t) & -x_2'(r) \end{vmatrix}^2} - \varepsilon N_2^2 \frac{\begin{vmatrix} y_1'(t) & x_2(r) - x_1(t) \\ x_1'(t) & y_2(r) - y_1(t) \end{vmatrix}^2}{\begin{vmatrix} y_1'(t) & -y_2'(r) \\ x_1'(t) & -x_2'(r) \end{vmatrix}^2} = 0. \tag{12}$$

Consequently, using (5) we have explicit equations of bisector curves as follows:

$$\begin{aligned} P_1(t,r) &= [y_1'^2(t) - x_1'^2(t)][(y_2(r) - y_1(t))y_2'(r) - (x_2(r) - x_1(t))x_2'(r)]^2 \\ &-[y_1'^2(t) - x_1'^2(t)][(y_2(r) - y_1(t))y_1'(t) - (x_2(r) - x_1(t))x_1'(t)]^2 = 0. \end{aligned} \tag{13}$$

and

$$P_2(t,r) = [y_1'^2(t) - x_1'^2(t)][(y_2(r) - y_1(t))y_2'(r) - (x_2(r) - x_1(t))x_2'(r)]^2$$
$$(14)\quad +[y_1'^2(t) - x_1'^2(t)][(y_2(r) - y_1(t))y_1'(t) - (x_2(r) - x_1(t))x_1'(t)]^2 = 0.$$

Both of (13) and (14) represent the bisector curves of $C_1(t)$ and $C_2(r)$ in Lorentzian plane.

3.2. **Function $\mathbb{F}_2(t,r)$.** In this method, First of all we obtain the bisector points $P(t,r)$. When a point $P(t,r)$ is on the bisector of two curves $C_1(s)$ and $C_2(t)$, there exist (at least) two points $C_1(s)$ and $C_2(t)$ such that point P is simultaneously contained in the normal lines $L_1(t)$ and $L_2(r)$. As a result, the point P satisfies the following two linear equations:

$$(15)\qquad \begin{cases} L_1(t): & < P - C_1(s), T_1(s) >= 0 \\ L_2(r): & < P - C_2(t), T_2(t) >= 0. \end{cases}$$

Direct computation shows that

$$(16)\qquad \begin{cases} < P, T_1(t) >=< C_1(t), T_1(t) > \\ < P, T_2(r) >=< C_2(r), T_2(r) > . \end{cases}$$

Substituting (3) and (4) into (16), then using Cramer's rule leads to the bisector point $P(t,r) = (x(t,r), y(t,r))$ obtained by

$$(17)\qquad x(t,r) = \frac{\begin{vmatrix} x_1(t)x_1'(t) - y_1(t)y_1'(t) & -y_1'(t) \\ x_2(r)x_2'(r) - y_2(r)y_2'(r) & -y_2'(r) \end{vmatrix}}{\begin{vmatrix} x_1'(t) & -y_1'(t) \\ x_2'(r) & -y_2'(r) \end{vmatrix}},$$

$$(18)\qquad y(t,r) = \frac{\begin{vmatrix} x_1'(t) & x_1(t)x_1'(t) - y_1(t)y_1'(t) \\ x_2'(r) & x_2(r)x_2'(r) - y_2(r)y_2'(r) \end{vmatrix}}{\begin{vmatrix} x_1'(t) & -y_1'(t) \\ x_2'(r) & -y_2'(r) \end{vmatrix}}.$$

In addition, point $P(t,r)$ must satisfy the equidistance condition given as follows

$$\|P(t,r) - C_1(t)\| = \|P(t,r) - C_2(r)\|.$$

From (2), it is easy to see that

$$|\langle P - C_1(t), P - C_1(t)\rangle| = |\langle P - C_2(r), P - C_2(r)\rangle|.$$

Since it is an absolute value equation, there are two cases to consider. Now, we distinguish the following two cases:

Case 1. If $\langle P - C_1(t), P - C_1(t)\rangle = \langle P - C_2(r), P - C_2(r)\rangle$, then the bisector $P(t,r)$ must satisfy the following condition given as follows

$$(19)\qquad < P(t,r), C_1(t) - C_2(r) > \;=\; \frac{C_1(t)^2 - C_2(r)^2}{2}.$$

Case2: If $\langle P - C_1(t), P - C_1(t)\rangle = -\langle P - C_2(r), P - C_2(r)\rangle$, then the bisector $P(t,r)$ must satisfy the following condition given as follows

$$(20)\qquad < P(t,r), P(t,r) - (C_1(t) + C_2(r)) > \;=\; -\frac{C_1(t)^2 + C_2(r)^2}{2}.$$

Consequently, both of (19) and (20) represent the bisector curve in Lorentzian plane.

4. Examples

We demonstrate the bisector curves by the following examples. Throughout this chapter, we assume that $P_1(t,r)$ and $P_2(t,r)$ are the solution of (19) and (20) plotted in red and blue in figures, respectively.

Example 1. Let us consider the bisector of two lines $C_1(t)$ and $C_2(r)$ in Lorentzian plane given by parametrization

$$C_1(t) = (2t-1, t), \qquad C_2(r) = (1, r). \tag{21}$$

From (17) and (18), we have

$$x(t,r) = \frac{3t-2+r}{2}, \qquad y(t,r) = r. \tag{22}$$

Using (19), (21) and (22), the bisector curve $P_1(t,r)$ is obtained as follows

$$P_1(t,r) = 3t^2 - 6t - 2r + r^2 + 4 = 0. \tag{23}$$

From (20), (21) and (22), we get the bisector curve $P_2(t,r)$ given by

$$P_2(t,r) = 3t^2 - 12t + 6tr + 8 - r^2 - 4r = 0. \tag{24}$$

Consequently, the equations (23) and (24) as the bisector of two lines in Lorentzian plane illustrated in Figure 1b. The bisector of two lines in Euclidean plane is also shown in Figure 1a.

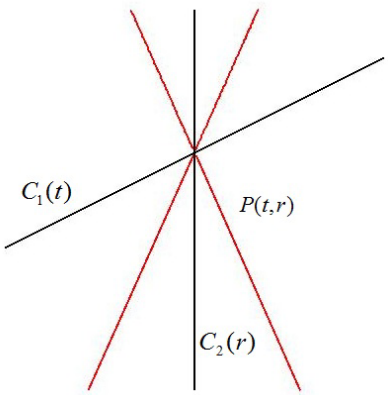

Figure 1a

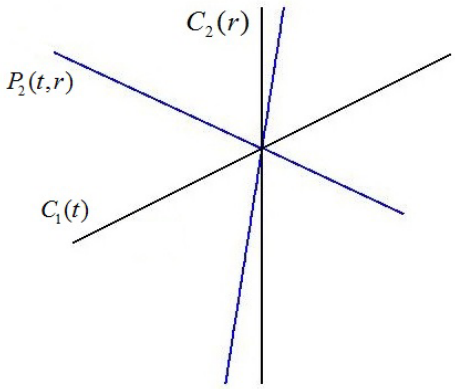

Figure 1b

Example 2. In this example, Figure 2a and Figure 2b show the bisector curve for a curve and a line give by parametrization

$$C_1(t) = (t^2, t), \qquad C_2(r) = (0, r)$$

The bisector curves are obtained by

$$P_1(t,r) = t^2 + 2rt^2 + r^2 - 3t^4 = 0$$

and $P_2(t,r) = t^2 - 5t^4 + 6t^2r + 8t^6 - 16t^4r + 8t^2r^2 - r^2 = 0$

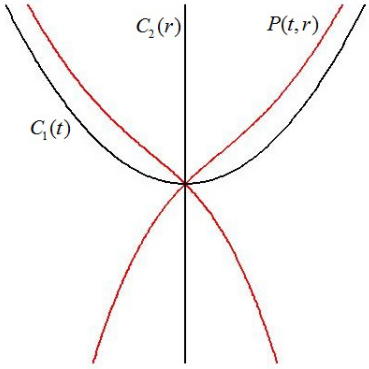

Figure 2a

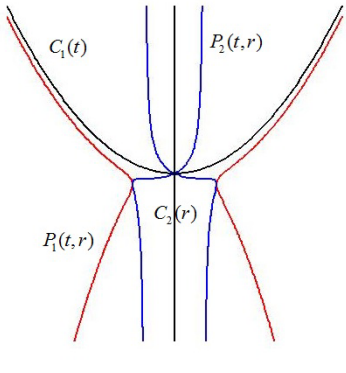

Figure 2b

Figure 3a and Figure 3b show bisector of a circle and line in Euclidean plane and Lorentzian plane, respectively.

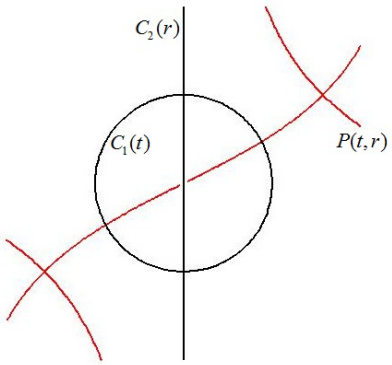

Figure 3a

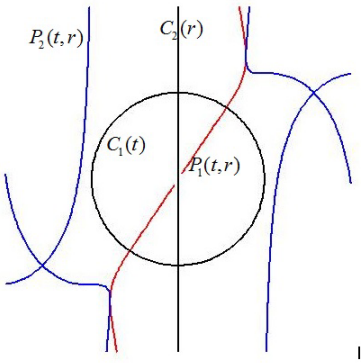

Figure 3b

Consequently, The bisector of two parabolas is illustrated in Figure 4a and Figure 4b in Euclidean plane and Lorentzian plane, respectively.

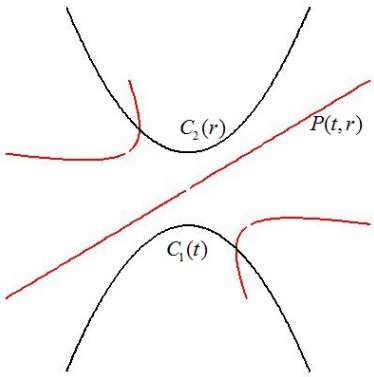

Figure 4a

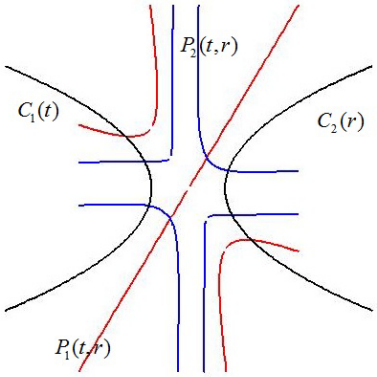

Figure 4b

5. Conclusion

This paper enlightens the role of Lorentzian geometry in the computation of bisector planar curves. It is also a collection of basic algorithms and linear constructions for that kind of curves. In the scope of this article, we have shown that the bisector between plane curves is a plane curve in Lorentzian plane. In a further contribution, we will study similar results for point-surface, sphere-surface and plane-surface bisectors in Lorentzian geometry.

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