A NEW PROPERTY OF
CIRCUMSCRIBED QUADRILATERAL

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Abstract. In this note we will give a new property of circumscribed quadrilateral.

1. Introduction

A circumscribed quadrilateral is a convex quadrilateral with an incircle, that is a circle tangent to all four sides. Figure 1 shows a circumscribed quadrilateral $ABCD$, where his incircle touch its sides $AB, BC, CD, DA$ at the points $X, Y, Z, T$, respectively.

Figure 1

Other names for these quadrilaterals are tangent quadrilateral, inscriptible quadrilateral and circumscribable quadrilateral. For more details we refer to the monograph of D. Grinberg [3] or D. Mihalca, I. Chişescu and M. Chişnă [9] and to the papers of T. Andreescu and B. Enescu [1], W. Chao and P. Simeonov [2], M. Josefsson [4], [5], [6], M. Hajja [7], L. Hoehn [8], N. Minculete [10] and M. De Villiers [11].

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A convex quadrilateral with the sides $a, b, c, d$ is tangential if and only if
\begin{equation}
    a + c = b + d
\end{equation}
according to the Pitot theorem [1, pp. 65-67].

The following result was obtained by A. Zaslavsky in [3].

In a circumscribed quadrilateral $ABCD$ we note with $K, L, M$ and $N$ the projections of the intersection point of the diagonals of $ABCD$ on the $[AB], [BC], [CD]$ and $[AD]$ sides. The following relation holds:
\begin{equation}
    \frac{1}{|OK|} + \frac{1}{|OM|} = \frac{1}{|OL|} + \frac{1}{|ON|}.
\end{equation}

2. Main result

**Theorem 2.1.** Let $ABCD$ be a circumscribed quadrilateral and $O$ is the point of intersection of its diagonals. Let $A_1B_1C_1D_1$ be a quadrilateral obtained by inversion of pole $O$ of quadrilateral $ABCD$. Then $A_1B_1C_1D_1$ is circumscribed quadrilateral.

**Proof.** Let $K, L, M$ and $N$ be the projections of the point $O$ on the $[AB], [BC], [CD]$ and $[AD]$ sides, respectively (see Figure 2).

![Figure 2](image-url)

Denote by $S_{OAB}$ the area of the triangle $OAB$ and by $k$ the ratio of the inversion of pole $O$. From the equalities
\begin{equation}
    2S_{OAB} = |OA| |OB| \left| \sin \overline{AOB} \right| = |AB| \cdot |OK|,
\end{equation}
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we obtain

\[ \frac{|AB|}{|OA| \cdot |OB|} = \frac{\sin \angle AOB}{|OK|} \]

Similarly, we have

\[ \frac{|CD|}{|OC| \cdot |OD|} = \frac{\sin \angle COD}{|OM|} \]

Because \( \sin \angle AOB = \sin \angle COD \), by (5) and (6) results

\[ |A_1B_1| + |C_1D_1| = k \cdot \left( \frac{|AB|}{|OA| \cdot |OB|} + \frac{|CD|}{|OC| \cdot |OD|} \right) \]

\[ = k \cdot \sin \angle AOB \cdot \left( \frac{1}{|OK|} + \frac{1}{|OM|} \right) \]

Similarly, we have

\[ |A_1D_1| + |B_1C_1| = k \cdot \sin \angle DOA \cdot \left( \frac{1}{|OL|} + \frac{1}{|ON|} \right) \]

Using the relation (2), we have

\[ \frac{1}{|OK|} + \frac{1}{|OM|} = \frac{1}{|OL|} + \frac{1}{|ON|} \]

Because \( \sin \angle AOB = \sin \angle DOA \), by (6), (7) and (8), we obtain that:

\[ |A_1B_1| + |C_1D_1| = |A_1D_1| + |B_1C_1| \]

Now, by (1) and (9) result the conclusion. \( \square \)

References


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