ON SOME WEIGHTED ERDÖS-MORDELL’S TYPE INEQUALITIES FOR POLYgons

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Abstract. In this paper, we establish a simple proof of the Wolstenholme cyclic inequality and generalized main result of the Bombardelli and Wu [1].

1. Introduction

Let $\triangle A_1A_2A_3$ be a triangle, and let $P$ be an interior point of $\triangle A_1A_2A_3$. We denote the distance from $P$ to the vertices by $PA_i = r_i$, $i = 1, 2, 3$, and the distances from $P$ to the sides $A_1A_2, A_2A_3, A_3A_1$, by $d_{1,2}, d_{2,3}, d_{3,1}$, respectively. The famous Erdös-Mordell inequality asserts that

$$r_1 + r_2 + r_3 \geq 2(d_{1,2} + d_{2,3} + d_{3,1})$$

with equality if and only if the triangle is equilateral and the point $P$ is its center.

The (1) inequality was proposed by Erdös [4] as a conjectured and proved by Mordell and Barrow [11]. Some related results with historical comments on this problem can be found in [2, 8, 10].

Let $P_n$ be a convex polygon with $n \geq 3$ vertices, and let $P$ be an interior point of $P_n$. The distances from $P$ to the sides $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1$ are denoted, respectively, by $d_{1,2}, d_{2,3}, \ldots, d_{n-1,n}, d_{n,1}$ and $w_{i,i+1}$ denote the length of the bisector of the angle $A_iPA_{i+1}$ from $P$ to its intersection with the side $A_iA_{i+1}$ ($i = 1, 2, \ldots, n, A_{n+1} = A_1$). The distances $PA_i$ are denoted by $r_i$, $i = 1, 2, \ldots, n$. We also denote $\theta_{i,i+1} = \angle A_iPA_{i+1}, i = 1, 2, \ldots, n$, where the index $i$ is taken modulo $n$.

Lenhard [9] established a remarkable inequality concerning the convex polygon as an extension of Erdös-Mordell inequality as follows

$$\sum_{i=1}^{n} r_i \geq \frac{1}{\cos \frac{\pi}{n}} \sum_{i=1}^{n} w_{i,i+1} \geq \frac{1}{\cos \frac{\pi}{n}} \sum_{i=1}^{n} d_{i,i+1}.$$ 

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Dar and Gueron [3] proved a weighted Erdős-Mordell inequality:

\[
\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 \geq 2 \left( \sqrt{\lambda_1 \lambda_2 d_{1,2}} + \sqrt{\lambda_2 \lambda_3 d_{2,3}} + \sqrt{\lambda_3 \lambda_1 d_{3,1}} \right)
\]

where \( \lambda_1, \lambda_2, \lambda_3 \) are positive numbers.

In a recent paper, Gueron and Shafrir [5] generalized (3) as follows:

\[
\sum_{i=1}^{n} \lambda_i r_i \geq \left( \sec \frac{\pi}{n} \right) \sum_{i=1}^{n} \sqrt{\lambda_i \lambda_{i+1} d_{i,i+1}}
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are positive numbers and \( \lambda_{n+1} = \lambda_1 \).

In 2004, Janous [7] generalized Dar-Gueron’s inequality by introducing an exponential parameter, as follows

\[
\lambda_1 r_1^t + \lambda_2 r_2^t + \lambda_3 r_3^t \geq 2^{\min\{t,1\}} \left( \sqrt{\lambda_1 \lambda_2 d_{1,2}^t} + \sqrt{\lambda_2 \lambda_3 d_{2,3}^t} + \sqrt{\lambda_3 \lambda_1 d_{3,1}^t} \right)
\]

where \( \lambda_1, \lambda_2, \lambda_3 \) and \( t \) are positive real numbers.

In a recent paper [13], Shanhe Wu sharpened Janous’s inequality in the following form:

\[
\lambda_1 r_1^t + \lambda_2 r_2^t + \lambda_3 r_3^t \geq 2^{\min\{t,1\}} \left( \sqrt{\lambda_1 \lambda_2 \omega_{1,2}^t} + \sqrt{\lambda_2 \lambda_3 \omega_{2,3}^t} + \sqrt{\lambda_3 \lambda_1 \omega_{3,1}^t} \right)
\]

where \( \lambda_1, \lambda_2, \lambda_3 \) and \( t \) are positive real numbers.

2. Lemmas

**Lemma 2.1.** For any positive \( x_1, x_2, \ldots, x_n \) we have,

\[
\sum_{i=1}^{n} x_i^2 \geq \frac{1}{\cos \frac{\pi}{n}} \sum_{i=1}^{n} x_i x_{i+1} \cos \left( \frac{\theta_{i,i+1}}{2} \right).
\]

**Proof.** In a coordinate plane we choose the points \( B_i(u_i, v_i), i = 1, n \) such that \( OB_i = x_i, \angle B_iOB_{i+1} = \frac{\theta_{i,i+1}}{2}, i = 1, n \), where the index \( i \) is taken modulo \( n \). Let \( B_i B_{i+1} = b_i, i = 1, n \). Apply the cosine theorem for each of the triangles \( B_iOB_{i+1}, i = 1, n \) we have

\[
b_i^2 = x_i^2 + x_{i+1}^2 - 2 x_i x_{i+1} \cos \frac{\theta_{i,i+1}}{2}.
\]

It follows that,

\[
x_i x_{i+1} \cos \frac{\theta_{i,i+1}}{2} = \frac{1}{2} (u_i^2 + v_i^2 + u_{i+1}^2 + v_{i+1}^2 - (u_{i+1} - u_i)^2 - (v_{i+1} - v_i)^2)
\]

\[= u_i u_{i+1} + v_i v_{i+1}, \quad \forall i \in \{1, 2, \ldots, n-1\}\]

and

\[
x_1 x_n \cos \frac{\theta_{1,n+1}}{2} = x_1 x_n \cos \left( \pi - \sum_{i=1}^{n-1} \frac{\theta_{i,i+1}}{2} \right)
\]

\[= -x_1 x_n \cos \left( \sum_{i=1}^{n-1} \frac{\theta_{i,i+1}}{2} \right)
\]

\[= \frac{1}{2} (x_1^2 + x_n^2 - b_n^2) = -u_1 u_n - v_1 v_n.
\]
Therefore,

\[
\cos \frac{\pi}{n} \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i x_{i+1} \cos \frac{\theta_{i+1}}{2} = \]

\[
\cos \pi \frac{n}{n} \sum_{i=1}^{n} (u_i^2 + v_i^3) - \sum_{i=1}^{n-1} (u_i u_{i+1} + v_i v_{i+1}) + u_1 u_n + v_1 v_n.
\]

It is suffices to prove following two inequality,

(4) \[ u_1 u_2 + \cdots + u_{n-1} u_n - u_1 u_n \leq \cos \frac{\pi}{n} (u_1^2 + u_2^2 + \cdots + u_n^2), \]

(5) \[ v_1 v_2 + \cdots + v_{n-1} v_n - v_1 v_n \leq \cos \frac{\pi}{n} (v_1^2 + v_2^2 + \cdots + v_n^2). \]

Now we will prove the inequality (4). By the same method the inequality (5) will be proved. First we observed that for any positive real numbers \(x_1, y_1\) following inequality holds

\[
\left( \sqrt{x_1} u_1 - \frac{1}{2\sqrt{x_1}} u_2 + \sqrt{y_1} u_n \right) \geq 0
\]

i.e.

\[
u_1 u_2 + \sqrt{y_1} u_2 u_n - 2\sqrt{x_1 y_1} u_1 u_n \leq x_1 u_1^2 + \frac{1}{4x_1} u_2^2 + y_1 u_n^2.
\]

By the same way for any positive real numbers \(x_2, y_2, \ldots, x_{n-2}, y_{n-2}\) we get that

\[
u_2 u_3 + \sqrt{y_2} u_2 u_3 - 2\sqrt{x_2 y_2} u_2 u_n \leq x_2 u_2^2 + \frac{1}{4x_2} u_3^2 + y_2 u_n^2
\]

\[
u_{n-3} u_{n-2} + \sqrt{y_{n-3}} u_{n-3} u_{n-2} - 2\sqrt{x_{n-3} y_{n-3}} u_{n-3} u_n \leq x_{n-3} u_{n-3}^2 + \frac{1}{4x_{n-3}} u_{n-2}^2 + y_{n-3} u_n^2
\]

(6) \[
u_{n-2} u_{n-1} + \sqrt{y_{n-2}} u_{n-2} u_{n-1} - 2\sqrt{x_{n-2} y_{n-2}} u_{n-2} u_n \leq x_{n-2} u_{n-2}^2 + \frac{1}{4x_{n-2}} u_{n-1}^2 + y_{n-2} u_n^2.
\]

We choose the numbers \(x_1, x_2, \ldots, x_{n-2}, y_1, y_2, \ldots, y_{n-2}\) such that

\[
x_1 y_1 = \frac{1}{4} \sqrt{y_1} x_1 = 2\sqrt{x_2 y_2}, \ldots, \sqrt{y_{n-3}} x_{n-3} = 2\sqrt{x_{n-2} y_{n-2}}, y_{n-2} = x_{n-2}
\]

and

(7) \[
x_1 = \frac{1}{4x_1} + x_2 = \cdots = \frac{1}{4x_{n-3}} + x_{n-2} = \frac{1}{4x_{n-2}} = y_1 + y_2 + \cdots + y_{n-2}.
\]

It is easy to see that \(0 < x_1 < 1\). Thus there exists \(\alpha \in (0, \frac{\pi}{2})\) such that \(x_1 = \cos \alpha\). From the recurrent equation of (7) we find the numbers \(x_2, \ldots, x_{n-2}\) as follows

\[
x_2 = \frac{\sin 3\alpha}{2 \sin 2\alpha}, \ldots, x_{n-2} = \frac{\sin (n-1)\alpha}{2 \sin (n-2)\alpha}.
\]

Hence

\[
x_1 = \frac{1}{4x_{n-2}} \Leftrightarrow \cos \alpha = \frac{\sin (n-2)\alpha}{2 \sin (n-1)\alpha}.
\]
From here we get that $\alpha = \frac{\pi}{n}$. Now we find the numbers $y_1, y_2, \ldots, y_{n-2}$:

\[
y_1 = \frac{1}{4x_1} = \frac{1}{4} \cos \frac{\pi}{n} = \frac{2\sin \frac{\pi}{n}}{2\sin \frac{2\pi}{n}},
\]
\[
y_2 = \frac{y_1}{4x_1x_2} = \frac{1}{4} \cdot \frac{1}{4} \cos \frac{2\pi}{n} = \frac{2\sin \frac{2\pi}{n}}{2\sin \frac{3\pi}{n}},
\]
\[
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\]
\[
y_{n-2} = \frac{\sin(n-1)\frac{\pi}{n}}{2\sin(n-2)\frac{\pi}{n}} = \frac{\sin \frac{\pi}{n}}{2\sin \frac{(n-2)\pi}{n} \sin \frac{(n-1)\pi}{n}}.
\]

Finally, summing all inequality in (6) we have

\[
u_1 u_2 + u_2 u_3 + \cdots + u_{n-1} u_n - u_1 u_n \leq x_1 \cdot (u_1^2 + u_2^2 + \cdots + u_n^2) = \frac{\cos \frac{\pi}{n}}{n} (u_1^2 + u_2^2 + \cdots + u_n^2)
\]

then (4) inequality is proved.

**Lemma 2.2** (see [6, 14]). If $a_i > 0$, $\lambda_i > 0$, $i = 1, 2, \ldots, n$, $0 < t \leq 1$ then

\[
\sum_{i=1}^{n} a_i^t \leq \left( \sum_{i=1}^{n} \lambda_i \right)^{1-t} \cdot \left( \sum_{i=1}^{n} \lambda_i a_i \right)^{t}.
\]

**Lemma 2.3.** Let $x_i > 0$, $0 < \varphi_i < \frac{\pi}{2}$ $(i = 1, 2, \ldots, n)$, $n \geq 3$ and $\sum_{i=1}^{n} \varphi_i = \pi$. Then for $t > 0$ we have,

\[
\sum_{i=1}^{n} x_i x_{i+1} \cos^t \varphi_i \leq \left( \frac{\cos \frac{\pi}{n}}{n} \right)^{\min\{t,1\}} \cdot \sum_{i=1}^{n} x_i^2.
\]

**Proof.** Case(I). Let $0 < t \leq 1$. Using Lemma 2.2 we get,

\[
\sum_{i=1}^{n} x_i x_{i+1} \cos^t \varphi_i \leq \left( \sum_{i=1}^{n} x_i x_{i+1} \right)^{1-t} \cdot \left( \sum_{i=1}^{n} x_i x_{i+1} \cos \varphi_i \right)^{t} \leq \left( x_1^2 + \cdots + x_n^2 \right)^{1-t} \cdot \left( \frac{\cos \frac{\pi}{n}}{n} \sum_{i=1}^{n} x_i^2 \right)^{t} = \left( \frac{\cos \frac{\pi}{n}}{n} \right)^{t} \cdot \sum_{i=1}^{n} x_i^2.
\]

Case(II). Let $t > 1$. Using Lemma 2.1 we get,

\[
\sum_{i=1}^{n} x_i x_{i+1} \cos^t \varphi_i < \sum_{i=1}^{n} x_i x_{i+1} \cos \varphi_i \leq \frac{\cos \frac{\pi}{n}}{n} \sum_{i=1}^{n} x_i^2.
\]

Lemma 2.3 is proved.
Lemma 2.4 (see [1]). Let $Q$ be an interior point of polygon $A_1A_2 \ldots A_n$, and let $\angle A_1QA_2 = 2\alpha_1, \angle A_2QA_3 = 2\alpha_2, \ldots, \angle A_nQA_1 = 2\alpha_n$. The bisectors of $\angle A_1QA_2, \angle A_2QA_3, \ldots, \angle A_nQA_1$ intersect respectively the circumcircles of $\triangle A_1QA_2, \triangle A_2QA_3, \ldots, \triangle A_nQA_1$ in the points $A'_1,A'_2,\ldots,A'_n$. Let $QA_i = r_i, QA'_i = l_{i,i+1}, i = 1, 2, \ldots, n$. Then, we have the following identities

$$l_{1,2} = \frac{r_1 + r_2}{2\cos \alpha_1}, l_{2,3} = \frac{r_2 + r_3}{2\cos \alpha_2}, \ldots, l_{n,1} = \frac{r_n + r_1}{2\cos \alpha_n}.$$ 

3. Main results

Theorem 3.1. Suppose $Q$ is an interior point of polygon $A_1A_2 \ldots A_n$ and conditions of Lemma 2.4 holds. Then for $\lambda_i > 0 (i = 1, 2, \ldots, n)$ and $t < 0$, we have the inequality

$$l_{i,i+1} \geq \left(\cos \frac{\pi}{n}\right)^{\min\{-t,1\}} \cdot \sum_{i=1}^n \lambda_i r_i^t \geq \sum_{i=1}^n \sqrt{\lambda_i \lambda_{i+1}} \cdot l_{i,i+1}^t.$$ 

Proof. From Lemma 2.4 that

$$\sum_{i=1}^n \sqrt{\lambda_i \lambda_{i+1}} \cdot l_{i,i+1}^t = \sum_{i=1}^n \sqrt{\lambda_i \lambda_{i+1}} \left(\frac{2\cos \alpha_i}{r_i + r_{i+1}}\right)^{-t}.$$ 

Since $-t > 0$, by using the arithmetic-geometric means inequality and the inequality given by Lemma 2.3, we obtain:

$$\sum_{i=1}^n \sqrt{\lambda_i \lambda_{i+1}} \left(\frac{2\cos \alpha_i}{r_i + r_{i+1}}\right)^{-t} \leq \sum_{i=1}^n \sqrt{\lambda_i \lambda_{i+1}} r_i^t r_{i+1}^{-t} \cos^{-t} \alpha_i \leq \left(\cos \frac{\pi}{n}\right)^{\min\{-t,1\}} \cdot \sum_{i=1}^n \lambda_i r_i^t.$$ 

Combining identity (9) and inequality (10) yields the inequality (8). \(\square\)

4. Application to Erdös-Mordell’s type inequalities of Lemma 2.1

Theorem 4.1 (see [5]). For any positive $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have,

$$\sum_{i=1}^n \lambda_i r_i \geq \frac{1}{\cos \frac{\pi}{n}} \cdot \sum_{i=1}^n \sqrt{\lambda_i \lambda_{i+1}} d_{i,i+1}.$$ 

Proof. Choose the numbers such that $x_i = \sqrt{\lambda_i} r_i, i = 1, n$. From the Lemma 2.1 we have,

$$\sum_{i=1}^n \lambda_i x_i \geq \frac{1}{\cos \frac{\pi}{n}} \cdot \sum_{i=1}^n \sqrt{\lambda_i \lambda_{i+1}} \sqrt{r_i d_{i,i+1}} \cos \frac{\theta_{i,i+1}}{2}.$$
and then apply the inequality
\[ \sqrt{r_{i+1}^r} \cos \frac{\theta_{i,i+1}}{2} \geq \omega_{i,i+1} \geq d_{i,i+1}, \quad (i = 1, n) \]
we have,
\[ \sum_{i=1}^{n} \lambda_i r_i \geq \frac{1}{\cos \frac{\pi}{n}} \cdot \sum_{i=1}^{n} \sqrt{r_{i+1}^r} d_{i,i+1}. \]

The proof of Theorem 4.1 is complete. \( \square \)

**Theorem 4.2** (see [13]). For any positive \( \lambda_1, \lambda_2, \ldots, \lambda_n \) we have,
\[ \sum_{i=1}^{n} \lambda_i r_i^{t_i} \geq \left( \cos \frac{\pi}{n} \right)^{-\min\{t,1\}} \cdot \sum_{i=1}^{n} \sqrt{\lambda_i \lambda_{i+1}} \cdot \omega_{i,i+1}. \]

**Proof.** Choose the numbers \( x_i, i = 1, n \) such that \( x_i = \sqrt{\lambda_i r_i^{t_i}}, i = 1, n. \)
From the Lemma 2.3 we have,
\[ \left( \cos \frac{\pi}{n} \right)^{-\min\{t,1\}} \cdot \sum_{i=1}^{n} \lambda_i r_i^{t_i} \geq \sum_{i=1}^{n} \sqrt{\lambda_i \lambda_{i+1}} \cdot (\sqrt{r_{i+1}^r} \cos \varphi_i)^{t_i} \]
and then apply the inequality
\[ \sqrt{r_{i+1}^r} \cos \varphi_i \geq \omega_{i,i+1} \geq d_{i,i+1}, \quad (i = 1, n) \]
we get,
\[ \left( \cos \frac{\pi}{n} \right)^{-\min\{t,1\}} \sum_{i=1}^{n} \lambda_i r_i^{t_i} \geq \sum_{i=1}^{n} \sqrt{\lambda_i \lambda_{i+1}} \cdot \omega_{i,i+1}. \]
This completes the proof of Theorem 4.2. \( \square \)

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