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THE BARYCENTERS WITH FIXED WEIGHTS OF SOME VARIABLE SIMPLEXES AND THEIR LOCI

ANAMARIA BOROŞ, CORNEL PINTEA AND ANCA ROŞIAN

Abstract. In this paper we deal with the locus of the barycenters $\alpha A + \beta B + \mu M$, $\alpha, \beta, \mu \in \mathbb{R}$, $\alpha + \beta + \mu = 1$ where A, B are fixed points and M is a variable point on a certain conic. When M runs over a circle the locus of $\alpha A + \beta B + \mu M$ is found in two different ways. Similar problems are being considered in higher dimensions.

1. Introduction

It is well-known that the locus of the centroid of a variable triangle with base and circumcircle fixed is a circle [1, p. 35], [3, p. 43], [6, p. 80]. This locus circle is obtained by translating the circumcircle with respect to a certain vector and contracting the resulting circle by means of a certain homothety. We are going to extend this fact in the following directions: we first replace the centroid with some arbitrary barycenter with fixed weights and release afterwards the dimension of the affine ambient space to some arbitrary value. We also replace the fixed circumcircle with a hyperquadric with unique center. In each case we identify a certain similarity which moves the given hyperquadric to the locus hyperquadric. The main ingredients we are using are the position vector $\vec{r}_X := \overrightarrow{RX}$ of a point X with respect to a given reference point R and the vector equation of a line [4, pp. 51-53], [5, p. 68] alongside the translations and homotheties of various Euclidean affine spaces [4, pp. 114-116].

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2. The locus of the barycenter $\alpha A + \beta B + \mu M$

Let us consider an affine Euclidean plane $\mathcal{E}_2$ and O, A, B three given points. We plan to identify a similarity of the plane which maps an arbitrary point $M \in \mathcal{E}_2$ to its barycenter $\alpha A+\beta B+\mu M$, where $\alpha, \beta, \mu \in \mathbb{R}$, $\alpha+\beta+\mu = 1$. If $\vec{a} \in \vec{\mathcal{E}_2}$ is a vector and $O \in \mathcal{E}_2$, $k \in \mathbb{R}^*$, then $T_{\vec{a}} : \mathcal{E}_2 \longrightarrow \mathcal{E}_2$ and $H_{O,k} : \mathcal{E}_2 \longrightarrow \mathcal{E}_2$ denote the translation of vector $\vec{a}$ and the homothety of center O and ratio k.

Theorem 2.1. *If $O, A, B \in \mathcal{E}_2$ are given points and $\alpha, \beta, \mu \in \mathbb{R}, \alpha+\beta+\mu = 1$, then*

$$\left(H_{Q,\mu} \circ T_{\overrightarrow{OQ}}\right)(M) = \alpha A + \beta B + \mu M \tag{1}$$

for all $M \in \mathcal{E}_2$, where $Q = \alpha A + \beta B + \mu O$.

Proof. Taking into account that

$$\overrightarrow{Q(H_{Q,\mu} \circ T_{\overrightarrow{OQ}})(M)} = \mu\overrightarrow{QT_{\overrightarrow{OQ}}(M)} = \mu\overrightarrow{T_{\overrightarrow{OQ}}(O)T_{\overrightarrow{OQ}}(M)} = \mu\overrightarrow{OM},$$

one gets successively:

$$\begin{aligned}
\alpha A+\beta B+\mu M = \left(H_{Q,\mu} \circ T_{\overrightarrow{OQ}}\right)(M) \quad &\Leftrightarrow \\
\alpha\overrightarrow{QA}+\beta\overrightarrow{QB}+\mu\overrightarrow{QM} = \overrightarrow{Q(H_{Q,\mu} \circ T_{\overrightarrow{OQ}})(M)} \quad &\Leftrightarrow \\
\alpha(\overrightarrow{QO}+\overrightarrow{OA}) + \beta(\overrightarrow{QO}+\overrightarrow{OB}) + \mu(\overrightarrow{QO}+\overrightarrow{OM}) = \mu\overrightarrow{OM} \quad &\Leftrightarrow \\
\overrightarrow{QO} + \alpha\overrightarrow{OA} + \beta\overrightarrow{OB} = \vec{0} \quad &\Leftrightarrow \\
\overrightarrow{QO} + \overrightarrow{OQ} = \vec{0} \quad &\Leftrightarrow
\end{aligned}$$

Since the latter equality is obvious, it follows that the stated equality (1) is now completely proved. □

Corollary 2.2. *Let $E \subset \mathcal{E}_2$ be an ellipse of center O. If $A, B \in E$ are fixed points and $M \in E$ is a variable point, then the locus of the barycenter $\alpha A + \beta B + \mu M$, $\alpha, \beta, \mu \in \mathbb{R}, \alpha + \beta + \mu = 1$ is the ellipse $\left(H_{Q,\mu} \circ T_{\overrightarrow{OQ}}\right)(E)$ whose center is $Q = \alpha A + \beta B + \mu O$.*

(see Figure 1).

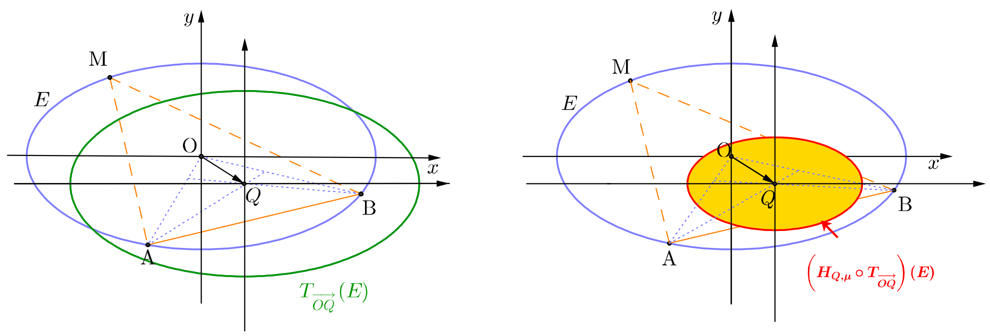
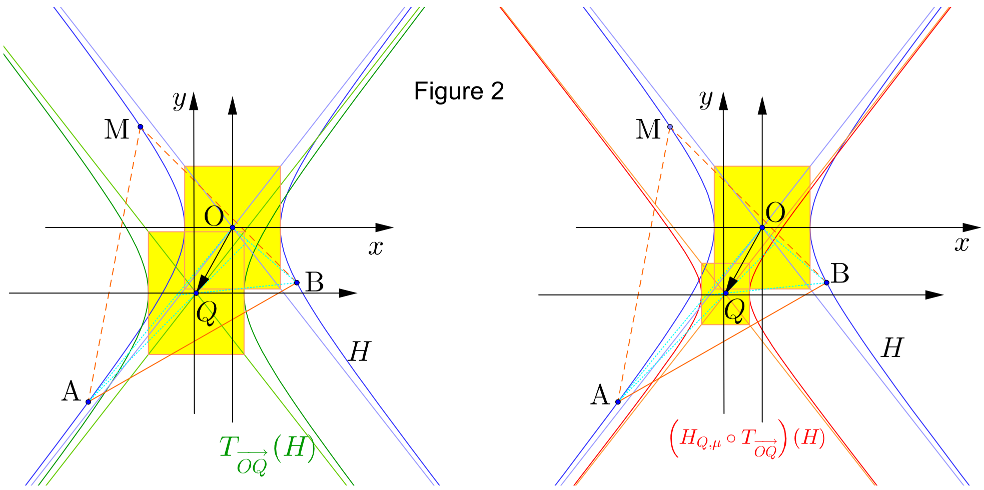


Figure 1

Corollary 2.3. *Let $H \subset \mathcal{E}_2$ be a hyperbola of center O. If $A, B \in H$ are fixed points and $M \in H$ is a variable point, then the locus of the barycenter $\alpha A + \beta B + \mu M$, $\alpha, \beta, \mu \in \mathbb{R}, \alpha + \beta + \mu = 1$ is the hyperbola $\left(H_{Q,\mu} \circ T_{\overrightarrow{OQ}}\right)(H)$ whose center is $Q = \alpha A + \beta B + \mu O$.*

(see Figure 2).



Lemma 2.4. Let $C \subset \mathcal{E}_2$ be an ellipse or hyperbola

$$C : \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.$$

If $A, B \in C$ are two fixed points and $M \in C$ is a variable point, then the locus of the barycenter $\alpha A + \beta B + \mu M$, $\alpha, \beta, \mu \in \mathbb{R}, \alpha + \beta + \mu = 1$ is the ellipse/hyperbola

$$C' : \frac{x'^2}{\mu^2 a^2} \pm \frac{y'^2}{\mu^2 b^2} = 1,$$

where

$$x' = x - \alpha x_A - \beta x_B$$
$$y' = y - \alpha x_A - \beta x_B.$$

In other words, the locus of the barycenter $\alpha A + \beta B + \mu M$, $\alpha, \beta, \mu \in \mathbb{R}$, $\alpha + \beta + \mu = 1$ is actually the conic with unique center at $\alpha A + \beta B + \mu O$, where O is the origin of the initial reference system, with parallel axes to those of the conic C and the length of semiaxes $a|\mu|, b|\mu|$.

Corollary 2.5. *Let $\mathcal{C}(O, R) \subset \mathcal{E}_2$ be a circle. If $A, B \in \mathcal{C}(O, R)$ are two fixed points and $M \in \mathcal{C}(O, R)$ is a variable point, then the locus of the barycenter $N := \alpha A + \beta B + \mu M, \alpha, \beta, \mu \in \mathbb{R}, \alpha + \beta + \mu = 1$ is the circle $\mathcal{C}(\alpha A + \beta B + \mu O, |\mu| R)$*

For Corollary 2.5 we can provide an alternative proof. In this respect we first prove the following:

Lemma 2.6. *If A, B, M are affine independent points in the affine plane $\mathcal{E}_2$ and $\alpha, \beta, \mu \in \mathbb{R}\backslash\{0,1\}$, $\alpha+\beta+\mu=1$, then $MN \cap AB = \left\{P := \frac{\alpha}{\alpha+\beta}A + \frac{\beta}{\alpha+\beta}B\right\}$, where N stands for the barycenter $\alpha A + \beta B + \mu M$. Moreover, the following relations hold*

$$\overrightarrow{NP} = \mu\, \overrightarrow{MP} \tag{2}$$

$$\vec{r}_M = \frac{1}{1-\alpha-\beta}\vec{r}_N + \frac{\alpha+\beta}{\alpha+\beta-1}\vec{r}_P. \tag{3}$$

Proof. The vector equations of the lines MN and AB are

$$\begin{aligned}
(MN) \quad & \vec{r}_X = \lambda \vec{r}_N + (1-\lambda)\vec{r}_M \\
& \vec{r}_X = \lambda(\alpha \vec{r}_A + \beta \vec{r}_B + \mu \vec{r}_M) + (1-\lambda)\vec{r}_M \\
& \vec{r}_X = \alpha\lambda \vec{r}_A + \beta\lambda \vec{r}_B + (\mu\lambda + 1 - \lambda)\vec{r}_M \\
(MN) \quad & \vec{r}_X = \alpha\lambda \vec{r}_A + \beta\lambda \vec{r}_B + (1 - \alpha\lambda - \beta\lambda)\vec{r}_M \\
(AB) \quad & \vec{r}_Y = (1-\nu)\vec{r}_A + \nu \vec{r}_B.
\end{aligned}$$

For $\lambda = \frac{1}{\alpha+\beta}$ and $\nu = \frac{\beta}{\alpha+\beta}$ one gets

$$\vec{r}_X = \frac{\alpha}{\alpha+\beta}\vec{r}_A + \frac{\beta}{\alpha+\beta}\vec{r}_B = \vec{r}_Y,$$

i.e the line MN passes through the point $P = \frac{\alpha}{\alpha+\beta}A + \frac{\beta}{\alpha+\beta}B \in AB$, namely $MN \cap AB = \{P\}$. Taking into account that

$$N = \alpha A + \beta B + \mu M \text{ and } P = \frac{\alpha}{\alpha+\beta}A + \frac{\beta}{\alpha+\beta}B,$$

one gets immediately

$$\overrightarrow{NM} = \alpha\overrightarrow{AM} + \beta\overrightarrow{BM} \text{ and } \overrightarrow{PM} = \frac{\alpha}{\alpha+\beta}\overrightarrow{AM} + \frac{\beta}{\alpha+\beta}\overrightarrow{BM},$$

which shows that $\overrightarrow{NM} = (\alpha+\beta)\overrightarrow{PM}$. Therefore we have successively:

$$\begin{aligned}
\overrightarrow{NM} = (\alpha+\beta)\overrightarrow{PM} \quad & \Leftrightarrow \quad \overrightarrow{NP} + \overrightarrow{PM} = (\alpha+\beta)\overrightarrow{PM} \\
& \Leftrightarrow \quad \overrightarrow{NP} = \overrightarrow{MP} - (\alpha+\beta)\overrightarrow{MP} \\
& \Leftrightarrow \quad \overrightarrow{NP} = \overrightarrow{MP} - (\alpha+\beta)\overrightarrow{MP} \\
& \Leftrightarrow \quad \overrightarrow{NP} = (1-\alpha-\beta)\overrightarrow{MP} \\
& \Leftrightarrow \quad \overrightarrow{NP} = \mu\overrightarrow{MP} \\
& \Leftrightarrow \quad \vec{r}_P - \vec{r}_N = \mu\,(\vec{r}_P - \vec{r}_M) \\
& \Leftrightarrow \quad \mu\vec{r}_M = \vec{r}_N + (\mu-1)\vec{r}_P \\
& \Leftrightarrow \quad \vec{r}_M = \frac{1}{\mu}\vec{r}_N + \frac{\mu-1}{\mu}\vec{r}_P \\
& \Leftrightarrow \quad \vec{r}_M = \frac{1}{1-\alpha-\beta}\vec{r}_N + \frac{\alpha+\beta}{\alpha+\beta-1}\vec{r}_P.
\end{aligned}$$

This completes the proof of the lemma. □

Remarks 2.7. Let $\mathcal{C}(O,R) \subset \mathcal{E}_2$ be a circle, let $A, B \in \mathcal{C}(O,R)$ be two fixed points and $M \in \mathcal{C}(O,R)$ be a variable point,

(1) The collection of lines MN, as M runs over the circle $\mathcal{C}(O,R)$ and N stands for the variable barycenter $N := \alpha A + \beta B + \mu M, \alpha, \beta, \mu \in (0,1)$, $\alpha + \beta + \mu = 1$, is a pencil of lines with the common point
$$P = \frac{\alpha}{\alpha+\beta}A + \frac{\beta}{\alpha+\beta}B.$$

(2) The locus of the incenter of the triangle ABM is a union of two circle arcs. Note that the incenter is a barycenter with the variable weights
$$\frac{a}{a+b+m}, \frac{b}{a+b+m} \text{ and } \frac{m}{a+b+m},$$
where $m = |AB|$, $a = |BM|$ and $b = |AM|$. We wonder whether the loci of some other remarkable barycenters, but with variable weights, such as Gergonne's point [2, p. 13] or Torricelli's point [6, p. 46, 61], associated to the triangle ABM are still circles or union of circle arcs, as M runs over the circle $\mathcal{C}(O,R)$.

Proof. (*Alternative proof for Corollary 2.5*) The relation (3) is equivalent to
$$(1-\alpha-\beta)(\vec{r}_M - \vec{r}_O) = \vec{r}_N - ((1-\alpha-\beta)\vec{r}_O + (\alpha+\beta)\vec{r}_P).$$
This shows that
$$\mu^2||(\vec{r}_M - \vec{r}_O)||^2 = ||\vec{r}_N - (\mu\vec{r}_O + (\alpha+\beta)\vec{r}_P)||^2$$
i.e.
$$||\vec{r}_N - (\mu\vec{r}_O + \alpha\vec{r}_A + \beta\vec{r}_B)||^2 = \mu^2R^2.$$
Thus $N \in \mathcal{C}(\alpha A + \beta B + \mu O, |\mu|R)$. □

3. The locus of a barycenter with fixed weights of a variable simplex

Let $\mathcal{E}_n$ be an n-dimensional affine Euclidean space and O, $A_1, \ldots, A_n$ be given points. As in the two dimensional case, we denote by $T_a : \mathcal{E}_n \longrightarrow \mathcal{E}_n$ and $H_{O,k} : \mathcal{E}_n \longrightarrow \mathcal{E}_n$ the translation of vector $a \in \vec{\mathcal{E}_n}$ and the homothety with center O and ratio $k \in \mathbb{R}^*$ respectively. We will identify here the suitable similarity of the space $\mathcal{E}_n$ which maps M to its barycenter $\alpha_1 A_1 + \cdots + \alpha_n A_n + \mu M$, where $\alpha_1, \ldots, \alpha_n, \mu \in \mathbb{R}$ are fixed scalars satisfying $\alpha_1 + \cdots + \alpha_n + \mu = 1$.

Theorem 3.1. *If O, $A_1, \ldots, A_n \in \mathcal{E}_n$ are given points and $\alpha_1, \ldots, \alpha_n, \mu \in \mathbb{R}$ are given scalars satisfying $\alpha_1 + \cdots + \alpha_n + \mu = 1$, then*
$$\left(H_{Q,\mu} \circ T_{\overrightarrow{OQ}}\right)(M) = \alpha_1 A_1 + \cdots + \alpha_n A_n + \mu M \tag{4}$$
for all $M \in \mathcal{E}_n$, where $Q = \alpha_1 A_1 + \cdots + \alpha_n A_n + \mu O$.

The proof of Theorem 3.1 works along the same lines as that of Theorem 2.1.

Corollary 3.2. *Let $\mathcal{H} \subset \mathcal{E}_n$ be a hyperquadric with unique center O. If $A_1, \ldots, A_n \in \mathcal{H}$ are n fixed points and $M \in \mathcal{H}$ is a variable point, then the locus of the barycenter $\alpha_1 A_1 + \cdots + \alpha_n A_n + \mu M$, $\alpha_1, \ldots, \alpha_n, \mu \in \mathbb{R}, \alpha_1 + \cdots + \alpha_n + \mu = 1$ is the hyperquadric $\left(H_{Q,\mu} \circ T_{\overrightarrow{OQ}} \right) (\mathcal{H})$ with unique center $Q = \alpha_1 A_1 + \cdots + \alpha_n A_n + \mu O$.*

Corollary 3.3. *Let $\mathcal{H} \subset \mathcal{E}_3$ be an ellipsoid or a hyperboloid of one or two sheets with center at O. If $A, B, C \in \mathcal{E}$ are fixed points and $M \in \mathcal{E}$ is a variable point, then the locus of the barycenter $\alpha A + \beta B + \gamma C + \mu M$, $\alpha, \beta, \gamma, \mu \in \mathbb{R}, \alpha + \beta + \gamma + \mu = 1$ is the the ellipsoid or the hyperboloid of one or two sheets respectively $\left(H_{Q,\mu} \circ T_{\overrightarrow{OQ}} \right) (\mathcal{H})$, where $Q = \alpha A + \beta B + \gamma C + \mu O$.*

Corollary 3.4. *Let $S(O, R) \subset \mathcal{E}_n$ be a hypersphere. If $A_1, \ldots, A_n \in \mathcal{S}(O, R)$ are fixed points and $M \in \mathcal{S}(O, R)$ is a variable point, then the locus of the barycenter $\alpha_1 A_1 + \cdots + \alpha_n A_n + \mu M$, $\alpha_1, \ldots, \alpha_n, \mu \in \mathbb{R}$, $\alpha_1 + \cdots + \alpha_n + \mu = 1$ of the simplex $A_1 \ldots A_n M$ is the hypersphere $S(\alpha_1 A_1 + \cdots + \alpha_n A_n + \mu O, |\mu| R)$. In particular, the locus of the centroid $G := \frac{1}{n+1} A_1 + \cdots + \frac{1}{n+1} A_n + \frac{1}{n+1} M$ is the hypersphere*

$$\mathcal{S} \left(\frac{1}{n+1} A_1 + \cdots + \frac{1}{n+1} A_n + \frac{1}{n+1} O, \frac{R}{n+1} \right).$$

Corollary 3.4 can be alternatively proved by means of the following:

Lemma 3.5. *If $A_1, \ldots, A_n, M \in \mathcal{E}_n$ are affine independent points and $\alpha_1, \cdots, \alpha_n, \mu \in \mathbb{R}$ are scalars such that $\alpha_1 + \cdots + \alpha_n + \mu = 1$, then*

$$MN \cap \text{aff}\{A_1, \ldots, A_n\} = \left\{ P := \frac{\alpha_1}{\alpha_1 + \cdots + \alpha_n} A_1 + \cdots + \frac{\alpha_n}{\alpha_1 + \cdots + \alpha_n} A_n \right\},$$

where N stands for the barycenter $\alpha_1 A_1 + \cdots + \alpha_n A_n + \mu M$. Moreover

$$\vec{r}_M = \frac{1}{1 - \alpha_1 - \cdots - \alpha_n} \vec{r}_N + \frac{\alpha_1 + \cdots + \alpha_n}{\alpha_1 + \cdots + \alpha_n - 1} \vec{r}_P. \tag{5}$$

The proof of Lemma 3.5 works along the same lines as the proof of Lemma 3.5.

Remarks 3.6. Let $S(O, R) \subset \mathcal{E}_n$ be a hypersphere. If $A_1, \ldots, A_n \in \mathcal{S}(O, R)$ are fixed points and $M \in \mathcal{S}(O, R)$ is a variable point, then the collection of lines MN as M runs over the sphere $S(O, R)$ and N stands for the variable barycenter $\alpha_1 A_1 + \cdots + \alpha_n A_n + \mu M$, $\alpha_1, \ldots, \alpha_n, \mu \in (0, 1)$, $\alpha_1 + \cdots + \alpha_n + \mu = 1$ is a pencil of lines with the common point

$$P = \frac{\alpha_1}{\alpha_1 + \cdots + \alpha_n} A_1 + \cdots + \frac{\alpha_n}{\alpha_1 + \cdots + \alpha_n} A_n.$$

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HIGH SCHOOL "RADU PETRESCU"
PRUNDU BÂRGAULUI 415, BISTRIŢA NĂSĂUD, ROMANIA
E-mail address: anamariaboros30@yahoo.com

"BABEŞ-BOLYAI" UNIVERSITY
MIHAIL KOGĂLNICEANU 1, 400084 CLUJ-NAPOCA, ROMANIA
E-mail address: cpintea@math.ubbcluj.ro

"BABEŞ-BOLYAI" UNIVERSITY
MIHAIL KOGĂLNICEANU 1, 400084 CLUJ-NAPOCA, ROMANIA
E-mail address: rosian_anca@yahoo.com