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SEVERAL GEOMETRIC INEQUALITIES OF ERDÖS - MORDELL TYPE IN THE CONVEX POLYGON

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Abstract. In this paper we present the several geometric inequalities of Erdös-Mordell type in the convex polygon, using the Cauchy Inequality.

1. Introduction

In [6], in colaboration with A. Gobej, we present some geometric inequalities of Erdös-Mordell type in the convex polygon. Here, we found others geometric inequalities of Erdös-Mordell type, using several known inequalities, in the convex polygon.

Let A_1,A_2,...,A_n the vertices of the convex polygon, $n \geq 3$, and M, a point interior to the polygon. We note with R_k the distances from M to the vertices A_k and we note with r_k the distances from M to the sides $[A_kA_{k+1}]$ of length $A_kA_{k+1} = a_k$, where $k = \overline{1,n}$ and $A_{n+1} \equiv A_1$. For all $k \in \{1, ..., n\}$ with $A_{n+1} \equiv A_1$ and $m\left(\widehat{A_kMA_{k+1}}\right) = \delta_k$ we have the following property:

$$\delta_1 + \delta_2 + ... + \delta_n = 2\pi.$$

L. Fejes Tóth conjectured a inequality which is refered to the convex polygon, recall in [1] şi [3], thus

$$\sum_{k=1}^{n} r_k \leq \cos\left(\frac{\pi}{n}\right) \sum_{k=1}^{n} R_k. \tag{1}$$

In 1961 H.-C. Lenhard proof the inequality (1), used the inequality

$$\sum_{k=1}^{n} w_k \leq \cos\left(\frac{\pi}{n}\right) \sum_{k=1}^{n} R_k, \tag{2}$$

which was established in [5], where w_k the length of the bisector of the angle A_kMA_{k+1}, $(\forall)$ $k = \overline{1,n}$ with $A_{n+1} \equiv A_1$.

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M. Dincă published other solution for inequality (1) in *Gazeta Matematică* Seria B in 1998 (see [4]). Another inequality of Erdös-Mordell type for convex polygon was given by N. Ozeki [9] in 1957, namely,

$$\prod_{k=1}^{n} R_k \geq \left(\sec\frac{\pi}{n}\right)^n \prod_{k=1}^{n} w_k \tag{3}$$

which proved the inequality 16.8 from [3] due to L. Fejes-Tóth, so

$$\prod_{k=1}^{n} R_k \geq \left(\sec\frac{\pi}{n}\right)^n \prod_{k=1}^{n} r_k. \tag{4}$$

R. R. Janić in [3], shows that in any convex polygon $A_1A_{2\ldots}A_n$, there is the inequality

$$\sum_{k=1}^{n} R_k \sin\frac{A_k}{2} \geq \sum_{k=1}^{n} r_k. \tag{5}$$

D. Buşneag proposed in GMB no. 1/1971 the problem 10876, which is an inequality of Erdös-Mordell type for convex polygon, thus,

$$\sum_{k=1}^{n} \frac{a_k}{r_k} \geq \frac{2p^2}{\Delta}, \tag{6}$$

where p is the semiperimeter of polygon $A_1A_{2\ldots}A_n$ and Δ is the area of polygon.

In connection with inequality (6), D. M. Bătineţu established [2] the inequality

$$\sum_{k=1}^{n} \frac{a_k}{r_k} \geq \frac{2p}{r} \tag{7}$$

if the polygon $A_1A_{2\ldots}A_n$ is circumscribed about a circle of radius r.

Among the relations established between the elements of polygon $A_1A_{2\ldots}A_n$ we can remark the following relation for Δ- the area of convex polygon $A_1A_{2\ldots}A_n$:

$$2\Delta = a_1r_1 + a_2r_2 + \ldots + a_nr_n. \tag{8}$$

We select several inequalities obtained from [6]:

$$R_k \geq \frac{r_{k-1} + r_k}{2\sin\frac{A_k}{2}} \tag{9}$$

hold for all $k \in \{1, 2, \ldots, n\}$, with $r_0 = r_n$,

$$\left(2\cos\frac{\pi}{n}\right)^n \prod_{k=1}^{n} R_k \geq \prod_{k=1}^{n} \left(r_{k-1} + r_k\right), \quad (r_0 = r_n) \tag{10}$$

and

$$\sum_{k=1}^{n} \frac{r_{k-1} + r_k}{R_k} \leq 2n\cos\frac{\pi}{n} \tag{11}$$

and

$$\sum_{k=1}^{n} \frac{R_k^2}{r_k} \geq \sec\frac{\pi}{n} \sum_{k=1}^{n} R_k. \tag{12}$$

2. Main results

First, we will follow some procedures used in paper [6], through which we will obtain some Erdös-Mordell-type inequalities for the convex polygon. Among these will apply the Cauchy Inequality

Theorem 2.1. *In any convex polygon $A_1A_{2...}A_n$, there is the inequality*

$$\sum_{k=1}^{n} \frac{r_k}{R_k + R_{k+1}} \leq 2n \cos \frac{\pi}{n}. \tag{13}$$

Proof. The inequality (11),

$$\sum_{k=1}^{n} \frac{r_{k-1} + r_k}{R_k} \leq 2n \cos \frac{\pi}{n},$$

with $r_0 = r_n$, is expanded in the following way,

$$\frac{r_n + r_1}{R_1} + \frac{r_1 + r_2}{R_2} + ... + \frac{r_{n-1} + r_n}{R_n} =$$

$$r_1 \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + r_2 \left(\frac{1}{R_2} + \frac{1}{R_3} \right) + ... + r_n \left(\frac{1}{R_n} + \frac{1}{R_1} \right) \leq 2n \cos \frac{\pi}{n}$$

On the other hand, we have

$$\frac{1}{R_{k-1}} + \frac{1}{R_k} \geq \frac{4}{R_{k-1} + R_k}, \ (\forall) \ k = \overline{1, n}$$

with $R_0 = R_n$, from where we can deduce another inequality, of an Erdös-Mordell type, namely,

$$\sum_{k=1}^{n} \frac{r_k}{R_k + R_{k+1}} \leq \frac{n}{2} \cos \frac{\pi}{n}.$$

□

Theorem 2.2. *For any convex polygon $A_1A_{2...}A_n$,we have the inequality*

$$\sum_{k=1}^{n} \frac{R_k + R_{k+1}}{r_k} \geq \frac{2n}{\cos \frac{n}{2}},$$

with $R_{n+1} = R_1$.

Proof. The inequality

$$\sum_{k=1}^{n} x_k y_k \cdot \sum_{k=1}^{n} \frac{x_k}{y_k} \geq \left(\sum_{k=1}^{n} x_k \right)^2$$

is well known, because it is a particulary case of Cauchy's inequality. In this we will take $x_k = \frac{r_k}{R_k + R_{k+1}}$ and $y_k = 1$. Thus, the inequality becomes

$$\sum_{k=1}^{n} \frac{r_k}{R_k + R_{k+1}} \cdot \sum_{k=1}^{n} \frac{R_k + R_{k+1}}{r_k} \geq n^2$$

and, if we use the inequality (13), we get

$$\sum_{k=1}^{n} \frac{R_k + R_{k+1}}{r_k} \geq \frac{2n}{\cos \frac{n}{2}},$$

with $R_{n+1} = R_1$. □

Theorem 2.3. *In any convex polygon $A_1A_{2...}A_n$, there is the inequality*

$$\sum_{k=1}^{n} \frac{\sqrt{r_{k-1}r_k}}{R_k} \leq n \cos \frac{\pi}{n}. \tag{15}$$

Proof. From Cauchy's inequality, we have

$$\sum_{k=1}^{n} x_k y_k \cdot \sum_{k=1}^{n} \frac{x_k}{y_k} \geq \left(\sum_{k=1}^{n} x_k \right)^2.$$

Using the substitutions

$$x_k = \frac{\sqrt{r_{k-1}r_k}}{R_k}$$

and $y_k = 1$, we deduce that the inequality

$$\sum_{k=1}^{n} \frac{\sqrt{r_{k-1}r_k}}{R_k} \cdot \sum_{k=1}^{n} \frac{R_k}{\sqrt{r_{k-1}r_k}} \geq n^2,$$

holds. However, from the relation (11), we obtain

$$\sum_{k=1}^{n} \frac{\sqrt{r_{k-1}r_k}}{R_k} \leq n \cos \frac{\pi}{n}$$

which implies the inequality

$$\sum_{k=1}^{n} \frac{R_k}{\sqrt{r_{k-1}r_k}} \geq \frac{n}{\cos \frac{\pi}{n}},$$

with $r_0 = r_n$. □

Remark 1. *The inequality (15) generalizes the problem 1045 of G. Tsinsifas from the magazine Crux Mathematicorum. This is also remarked in [7].*

Theorem 2.4. *In any convex polygon $A_1A_{2...}A_n$ there is the inequality*

$$\sum_{k=1}^{n} \frac{R_k^2}{\sqrt{r_{k-1}r_k}} \geq \frac{n}{\cos^2 \frac{\pi}{n}}, \tag{16}$$

with $r_0 = r_n$.

Proof. In the inequality

$$\sum_{k=1}^{n} x_k y_k \cdot \sum_{k=1}^{n} \frac{x_k}{y_k} \geq \left(\sum_{k=1}^{n} x_k \right)^2$$

we replaced $x_k = y_k = \frac{R_k}{\sqrt{r_{k-1}r_k}}$ and the inequality becomes

$$n \sum_{k=1}^{n} \frac{R_k^2}{\sqrt{r_{k-1}r_k}} \geq \left(\sum_{k=1}^{n} \frac{R_k}{\sqrt{r_{k-1}r_k}} \right)^2 \geq \frac{n^2}{\cos^2 \frac{\pi}{n}}.$$

This means that inequality (16) is true. □

Theorem 2.5. *In any convex polygon $A_1A_{2...}A_n$ there is the inequality*

$$\sum_{k=1}^{n} \frac{a_k r_k}{R_k R_{k+1}} \leq n \sin \frac{2\pi}{n},$$

with $A_{n+1} = A_1$.

Proof. We can be written the area of triangle A_kMA_{k+1} in two ways, thus

$$\frac{a_k r_k}{2} = \frac{R_k R_{k+1} \sin A_k M A_{k+1}}{2},$$

which implies the relation

$$\frac{a_k r_k}{R_k R_{k+1}} = \sin A_k M A_{k+1}$$

so, by passing to the sum, we get the relation

$$\sum_{k=1}^{n} \frac{a_k r_k}{R_k R_{k+1}} = \sum_{k=1}^{n} \sin A_k M A_{k+1},$$

with $A_{n+1} = A_1$. Because the function $f : (0, \infty) \to R$, is defined as $f(x) = \sin x$, is concave, we will apply the inequality Jensen, thus

$$\frac{1}{n} \sum_{k=1}^{n} \sin A_k M A_{k+1} \leq \sin \frac{1}{n} \left(\sum_{k=1}^{n} A_k M A_{k+1} \right) = \sin \frac{2\pi}{n}.$$

Therefore, we have

$$\sum_{k=1}^{n} A_k M A_{k+1} \leq n \sin \frac{2\pi}{n}.$$

Consequently, we obtain the inequality of the statement. □

Remark 2. *The equality hold in the above mentioned theorems when the polygon is regular.*

Remark 3. *On the a hand, we have the equality* (8),

$$2\Delta = a_1 r_1 + a_2 r_2 + ... + a_n r_n = \sum_{k=1}^{n} a_k r_k,$$

and on the other hand, we have the inequality Cauchy, where we will replace $x_k = \sqrt{a_k r_k}$ and $y_k = \sqrt{\frac{a_k}{r_k}}$,then

$$2\Delta \sum_{k=1}^{n} \frac{a_k}{r_k} = \sum_{k=1}^{n} a_k r_k \sum_{k=1}^{n} \frac{a_k}{r_k} \geq \left(\sum_{k=1}^{n} a_k \right)^2 = 4p^2$$

which proves the inequality (6).

Theorem 2.6. *In any convex polygon $A_1A_{2...}A_n$ there is the inequality*

$$\sum_{k=1}^{n} R_k^2 \sin \frac{A_k}{2} \geq \frac{\sec \frac{\pi}{n}}{n} \left(\sum_{k=1}^{n} r_k \right)^2 \tag{18}$$

holds.

Proof. Applying inequality (9), we have the inequality

$$R_k = MA_k \geq \frac{r_{k-1} + r_k}{2\sin\frac{A_k}{2}} \ (\forall) \ k = \overline{1,n}$$

with $r_0 = r_n$, and this, by squaring, becomes

$$4R_k^2 \sin\frac{A_k}{2} \geq \frac{(r_{k-1} + r_k)^2}{\sin\frac{A_k}{2}}$$

and taking the sum, we deduce

$$4\sum_{k=1}^{n} R_k^2 \sin\frac{A_k}{2} \geq \sum_{k=1}^{n} \frac{(r_{k-1} + r_k)^2}{\sin\frac{A_k}{2}} \geq \frac{\left[\sum\limits_{k=1}^{n} (r_{k-1} + r_k)^2\right]}{\sum\limits_{k=1}^{n} \sin\frac{A_k}{2}} \geq \frac{4\left(\sum\limits_{k=1}^{n} r_k\right)^2}{n\cos\frac{\pi}{n}}$$

so, we found inequality (18). □

Theorem 2.7. *In any convex polygon $A_1A_{2...}A_n$ there is the inequality*

$$\sum_{k=1}^{n} (R_k + R_{k+1})\, r_k \geq \frac{2\sec\frac{\pi}{n}}{n}\left(\sum_{k=1}^{n} r_k\right)^2 \tag{19}$$

holds.

Proof. From inequality (9), we have

$$R_k = MA_k \geq \frac{r_{k-1} + r_k}{2\sin\frac{A_k}{2}}, \ (\forall) \ k = \overline{1,n}$$

with $r_0 = r_n$, and this, by multiply with $(r_{k-1} + r_k)$,becomes

$$2\,(r_{k-1} + r_k)\, R_k \geq \frac{(r_{k-1} + r_k)^2}{\sin\frac{A_k}{2}}$$

and by passing to the sum, we obtain the relation

$$2\sum_{k=1}^{n} (r_{k-1} + r_k)\, R_k \geq \sum_{k=1}^{n} \frac{(r_{k-1} + r_k)^2}{\sin\frac{A_k}{2}} \geq \frac{\left[\sum\limits_{k=1}^{n} (r_{k-1} + r_k)\right]^2}{\sum\limits_{k=1}^{n} \sin\frac{A_k}{2}} \geq \frac{4\left(\sum\limits_{k=1}^{n} r_k\right)^2}{n\cos\frac{\pi}{n}}.$$

Therefore, we have

$$\sum_{k=1}^{n} (r_{k-1} + r_k)\, R_k \geq \frac{2\sec\frac{\pi}{n}}{n}\left(\sum_{k=1}^{n} r_k\right)^2.$$

But, it follows that

$$\sum_{k=1}^{n} (R_k + R_{k+1})\, r_k = \sum_{k=1}^{n} (r_{k-1} + r_k)\, R_k$$

which means that, we obtain the inequality of statement. □

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